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On the Motion of Circular Cylinders in a Viscous Fluid

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III. *On the Motion of Circular Cylinders in a Viscous Fluid.*

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Introduction.

The flow of viscous fluids has been dealt with in numerous mathematical researches. Unlike most other branches of theoretical physics, rational hydrodynamics entails complications due to the non-linearity of the equations of motion, and—in comparison with the effort expended—relatively inconsiderable advance has been made.

In the standard "slow motion" treatment the product, or inertia, terms of the equations are neglected, by means of which artifice some of the outstanding difficulties are conveniently circumvented. However, even with this simplification, comparatively few solutions to problems have been published. The need for a wider range of results of this particular type became apparent in connection with an ulterior investigation, which will be specified shortly. The present paper* is preliminary in this sense, that it deals mainly with such "slow-motion" problems, and only briefly with the extensions in view. On the other hand, the results, even in the immature stage, need not be without some immediate application, notably to problems relating to fluids of great viscosity. Solutions of this nature admit a further useful interpretation in connection with the deflection of an unloaded flat plate—but the application to elastic theory is outside the range of the paper.

The investigation is restricted to two-dimensional problems in which the fluid is supposed bounded by circular cylinders. The methods, which are believed to be in great measure novel, have also been found powerful in researches on more complex forms of boundary. The early paragraphs are devoted to an indication of a functional treatment, which, although tentative, has the advantages that it narrows the field of enquiry, and exposes properties that are fundamental in the desired solution. An illustration deals with a type of flow maintained between a pair of concentric cylinders, and special interest attaches to the case where the outer radius increases indefinitely. Here the stream-function determines the conditions of flow due to a stationary cylinder immersed in a uniform infinite stream, and develops the well-known anomalous characteristics associated with such types of motion by Stokes.†

The main portion of the paper deals with systems in which the finite boundaries are

* Permission to communicate the results was kindly granted by the Aeronautical Research Committee.

† 'Mathematical and Physical Papers,' vol. 3, p. 62.

 [Published June 26, 1925.

defined as a pair of mutually external circular cylinders. The method employed is an adaptation of STOKES' principle of successive reflections,* and leads to an expression of very general form, applicable to problems in which arbitrary distributions of velocity are prescribed over the cylinders. However, in the applications considered, attention is confined to the practical supposition that the boundaries are maintained in steady rotation.

The rotation problem has already been treated *in extenso* by JEFFERY† for the case where one cylinder wholly encloses the other. With reference to the external case JEFFERY states that "there is, in general, no steady motion of the fluid for which the velocity of the fluid vanishes at infinity," and presents, in illustration, the analysis for equal cylinders spinning with equal angular velocities in opposite senses. He adds that the more general problem entails some complication. In the present paper the complete solution for the external case is obtained, but the greater part of the intermediary reductions have been omitted. It is believed that the results do not encroach to any important degree upon JEFFERY'S published work.

Special consideration is given to the conditions of the stream at infinity. It is shown that a solution of normal character may be obtained consistent with given rates of rotation of the cylinders and with constancy of the stream at infinity; but that the *magnitude* of that stream devolves upon the other conditions. Further, that a possible state of steady motion in a fluid at rest at infinity is where any pair of spinning cylinders are rotated appropriately as a "planetary" system about a particular "focus" situated on their line of centres.

In illustration of the more general results, a number of relatively simple problems, for which the solution reduces to finite terms, are treated in detail. The types examined include the simpler "planetary" systems, and the case in which a cylinder both rotates and translates in the presence of a fixed rigid wall. A calculation of the stress components and forces operating on the latter system shows that the viscous drag and the couple on the cylinder are, respectively, independent of the rotary and the translational motions. In the limiting case in which the cylinder approaches indefinitely close to the wall, both the drag and the couple tend to infinite values, and a parabolic distribution of velocity develops across the film of contact.

Some reference is now desirable to the particular application for which the results were intended. The complete equations of flow have been attacked by approximate methods from several angles. It will be sufficient here to cite the recent researches of BAIRSTOW,‡ CAVE AND LANG on the resistance of cylinders and plates. HARRISON'S§ paper on the motion of spheres and cylinders includes a valuable résumé of various methods in use. In general, these methods devolve upon an elaboration of the well-known approximation proposed by OSEEN.

* *Ibid.*, vol. 1, p. 28.

† 'Roy. Soc. Proc.,' A, vol. 101, pp. 169-174 (1922).

‡ 'Phil Trans. Royal Society,' A, vol. 223, pp. 383-432 (1923).

§ 'Trans. Camb. Phil. Soc.,' vol. 23, No. 4, pp. 71-88.

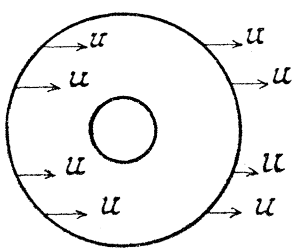
A different line of approach has been suggested by Cowley and Levy.* It entails the somewhat daring assumption that the stream function is expansible as an infinite series of positive powers of a non-dimensional parameter (Reynolds' number) UL/ν. The initial term of the series is the "slow-motion" solution, and the remaining functions are derived successively by integration of subsidiary linear equations. So far as the writer is aware, no adequate test of the utility of the method has as yet been made. In their original paper Cowley and Levy treat the case of laminar flow between rotating concentric cylinders as an instance of the efficiency of their mode of solution: but the illustration is inadequate, since the UL/ν expansion here clearly degenerates into the initial term.

In the absence of any convincing proof as to the legitimacy of the operations proposed, the most obvious procedure appeared to be an examination of the form of the expansion for a particular case. A difficulty, apparent at the outset, was the limited selection of problems for which the initial "slow-motion" term was already available. The present paper is intended to amplify the stock of preliminary results, thus preparing the ground for the more elaborate investigation.

The final section provides a sketch of the procedure and includes an illustrative application, carried as far as the first power of UL/ν, for the case of a single translating cylinder. The result, at this early stage of the expansion, cannot be viewed as promising, since it postulates the formation of eddies both behind and in *front of* the cylinder. On the other hand, these inadmissible conditions need not necessarily persist as the expansion proceeds, and do not *per se* constitute evidence that the method should be condemned. It is, however, well to remember that in this particular problem the analytical difficulties are accentuated by the anomalous form of the initial stream-function. With simple initial solutions of normal type, the UL/ν expansion might more readily be pursued to the higher powers, and the results be found more convincing.

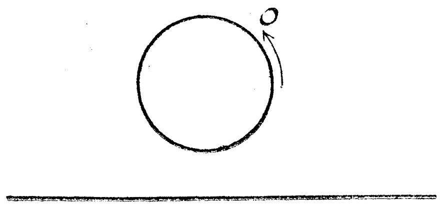
The following Contents Table summarises the range of the paper. In the explanatory diagrams the boundaries are indicated in heavy lines, and their prescribed state of motion by arrows. In cases where a boundary is *fixed* no symbol is appended.

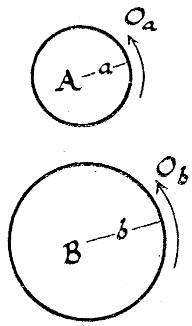
CONTENTS.

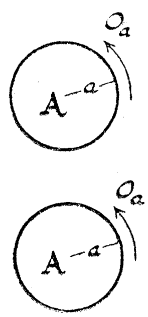


The radius of the outer circular boundary is eventually increased indefinitely.

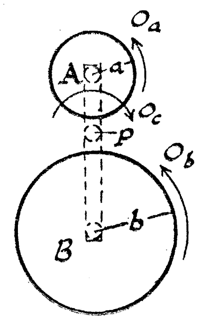
* 'Phil. Mag.,' vol. 41, pp. 584-607 (1921).



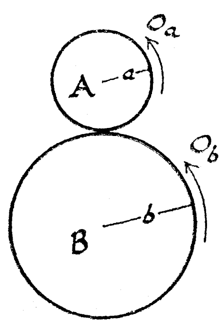


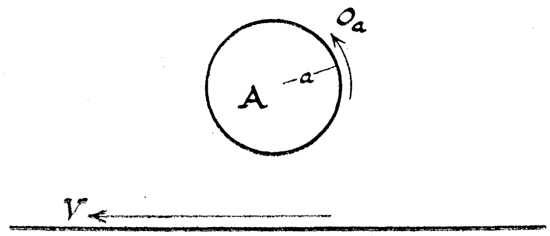


Equal radii and equal angular velocities.



A and B rotate about their centres, and the entire system rotates about P.





§ 1. *The Equations of Motion.*

The paper is restricted to steady two-dimensional flow, and deals primarily with solutions where inertia terms are viewed as negligible. For convenience, the equations for the stream function are initially quoted below in their general form, inclusive of product terms. The standard notation, as laid down by LAMB,* is adopted.

$$\nu\,\nabla^2\zeta = \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} \quad . \quad . \quad . \quad (1)$$

$$\zeta = \nabla^2\psi \quad . \quad . \quad . \quad (2)$$

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \quad . \quad . \quad . \quad (3)$$

In the sequel, extensive use will be made of the transformation

$$\lambda = x + iy, \quad \mu = x - iy, \quad . \quad . \quad . \quad (4)$$

in which

$$i \equiv \sqrt{-1}.$$

Here

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial\lambda\,\partial\mu},$$

and the equations may be replaced by the equivalent set

$$2\nu\,\frac{\partial^2\zeta}{\partial\lambda\,\partial\mu} = -i\left(\frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda}\right), \quad . \quad . \quad . \quad (5)$$

$$\zeta = 4\,\frac{\partial^2\psi}{\partial\lambda\,\partial\mu}, \quad . \quad . \quad . \quad (6)$$

$$u + iv = 2i\,\frac{\partial\psi}{\partial\mu}, \quad u - iv = -2i\,\frac{\partial\psi}{\partial\lambda} \quad . \quad . \quad . \quad (7)$$

* 'Treatise on Hydrodynamics,' Cambridge, 1916; p. 574.

For slow steady motion of the particular type considered, the equation for the stream function becomes

$$\frac{\partial^4 \psi}{\partial \lambda^2 \, \partial \mu^2} = 0, \qquad (8)$$

of which the most general integral is

$$8\psi = i\left[\mu F_1(\lambda) - \lambda F_2(\mu)\right] + F_3(\lambda) + F_4(\mu), \qquad (9)$$

F_1, F_2, etc., denoting arbitrary functions.

For motion under no external forces the equivalent mean-pressure function becomes

$$\frac{p}{\rho} = \frac{\nu}{2}\left[F'_1(\lambda) + F'_2(\mu)\right]. \qquad (10)$$

The following further formulæ conveniently determine the stress components.

$$p_{xx} + p_{yy} = -2p, \qquad (11\text{A})$$

$$p_{xx} - p_{yy} = -4i\nu\rho\left(\frac{\partial^2 \psi}{\partial \lambda^2} - \frac{\partial^2 \psi}{\partial \mu^2}\right); \qquad (11\text{B})$$

$$p_{xy} = p_{yx} = 2\nu\rho\left(\frac{\partial^2 \psi}{\partial \lambda^2} + \frac{\partial^2 \psi}{\partial \mu^2}\right). \qquad (11\text{C})$$

The major portion of the paper deals with functions of type (9), in relation to particular boundary requirements.

§ 2. *Single Translating Circular Cylinder.*

A functional method of solution has been found effective in the treatment of certain problems. The present paragraph provides a simple illustration.

The flow under consideration is imagined to take place in the annulus bounded by a pair of concentric cylinders. At each point of the outer cylinder the stream is supposed maintained uniform: whereas the inner boundary remains at rest (*see* diagram, contents table). The limiting case where the outer radius is increased indefinitely is interpreted as determining the flow due to a stationary cylinder immersed in a uniform stream (or, alternatively, the flow due to a single translating cylinder).

The boundary conditions are specified more precisely below, in the notation of §1. For convenience, the inner cylinder is selected of unit radius.

$$u = v = 0 \quad \text{for} \quad \lambda\mu = 1, \qquad (12\text{A})$$

$$u = U \quad \text{and} \quad v = 0 \quad \text{for} \quad \lambda\mu = R^2. \qquad (12\text{B})$$

Since the flow is clearly symmetrical about the axis $y = 0$, the appropriate integral form, equivalent to (9), will be

$$-2i\psi = [\lambda F(\mu) - \mu F(\lambda)] + f(\lambda) - f(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which F, f, denote functions to be determined.

The corresponding velocity components at any point become (*see* 7)

$$u + iv = F(\lambda) - \lambda F'(\mu) + f'(\mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (14A)$$

$$u - iv = F(\mu) - \mu F'(\lambda) + f'(\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad (14B)$$

Conditions (12) are satisfied provided the following functional relations are valid for all values of a parameter t.

$$0 = F\left(\frac{1}{t}\right) - \frac{1}{t} F'(t) + f'(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad (15A)$$

$$U = F\left(\frac{R^2}{t}\right) - \frac{R^2}{t} F'(t) + f'(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (15B)$$

Elimination of $f'(t)$ leads to the equation

$$U = F\left(\frac{R^2}{t}\right) - F\left(\frac{1}{t}\right) - (R^2 - 1)\frac{F'(t)}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Inspection suggests a solution of form

$$F(t) = \alpha t^2 + \beta \log t,$$

and, on substitution in (16), we obtain

$$\alpha = \frac{\beta}{(R^2 + 1)} = \frac{U}{2[(R^2 + 1)\log R - (R^2 - 1)]}.$$

The pressure is single-valued, and the stream function reduces to

$$-2i\psi = \alpha(\lambda - \mu)\left[(R^2 + 1)\log(\lambda\mu) - \lambda\mu + \frac{R^2}{\lambda\mu} - (R^2 - 1)\right]. \quad . \quad . \quad (17)$$

In polar co-ordinates the result may be expressed more generally as follows :

$$\psi = U \sin\theta \frac{\left[2(R_0^2 + R^2) r \log r + \frac{R_0^2 R^2}{r} - r^3 + \{R_0^2 - R^2 - 2(R_0^2 + R^2)\log R_0\} r\right]}{2\left[(R_0^2 + R^2)\log\left(\frac{R_0}{R}\right) - (R_0^2 - R^2)\right]}, \quad (18)$$

where

$$u = v = 0, \quad \text{for} \quad r = R_0,$$

$$u = U \quad \text{and} \quad v = 0, \quad \text{for} \quad r = R.$$

Representative stream-lines are shown plotted in fig. 1 for the special case $R_0 = 1$, $R = 5$, curves being drawn for constant values of $-\frac{36\psi}{U}$, where

$$-\frac{36\psi}{U} = \left(52r \log r + \frac{25}{r} - r^3 + 24r\right) \sin \theta. \quad . \quad . \quad . \quad (18A)$$

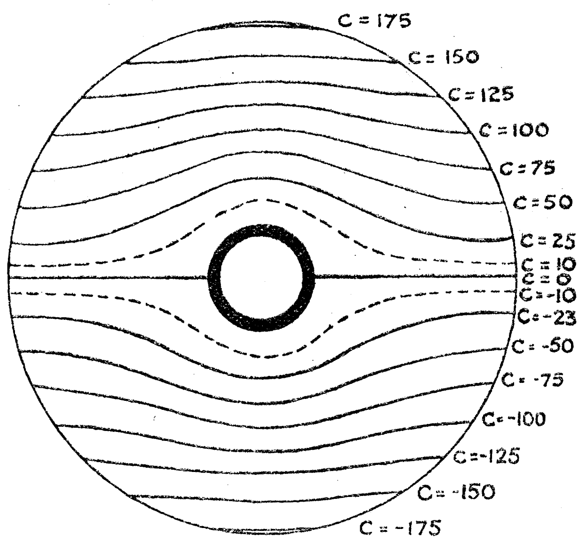


FIG. 1.—Slow steady flow is imagined to proceed in the annulus bounded by a pair of concentric circular cylinders $R = 5$ and $R_0 = 1$. At each point of the outer boundary $u = U$ (constant), and $v = 0$. The inner cylinder is fixed. Curves are drawn for constant values c of $-36\psi/U$ (*see* equation 18A of text).

When R is exceedingly large in comparison with the inner radius, the only term of importance that need be retained in (18) is

$$\frac{\psi}{U} = -\frac{\sin \theta \, r \log r}{\log R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This is a legitimate, but anomalous, form of stream function, which determines a uniform stream at great distances and zero velocity throughout the finite domain.* For fig. 2 representative curves, corresponding to (19), have been computed on the assumption that $\log_e R = 5$ (*i.e.*, $R = 148 \cdot 4$). Since the outer radius is now only moderately large—a convenience in the preparation of the diagram—the boundary conditions will only be satisfied approximately. Thus, whereas no radial component of velocity is admitted over the inner cylinder, there exists a maximum slipping component amounting to as much as $0 \cdot 2$ U. Discrepancies of a comparable order are allowed over the other boundary. Clearly, if R be chosen sufficiently large, the discrepancies will lie within any specified limits.

* In the solution proposed by BERRY and SWAIN anomalous characteristics are absent, but the velocity is logarithmically infinite at infinite distances; *see* 'Roy. Soc. Proc.,' A, vol. 102 (1923).

As was inferred by STOKES,* the anomalous form of (19) will persist with fixed internal boundaries of any cross-sectional shape which nowhere extend to infinity. This is evident from the form of the function (19), which is independent of the *shape* of the inner cylinder.

An enquiry which naturally suggests itself in this connection, is the search for simple stream functions in which anomalous features are absent. The most obvious modifica-

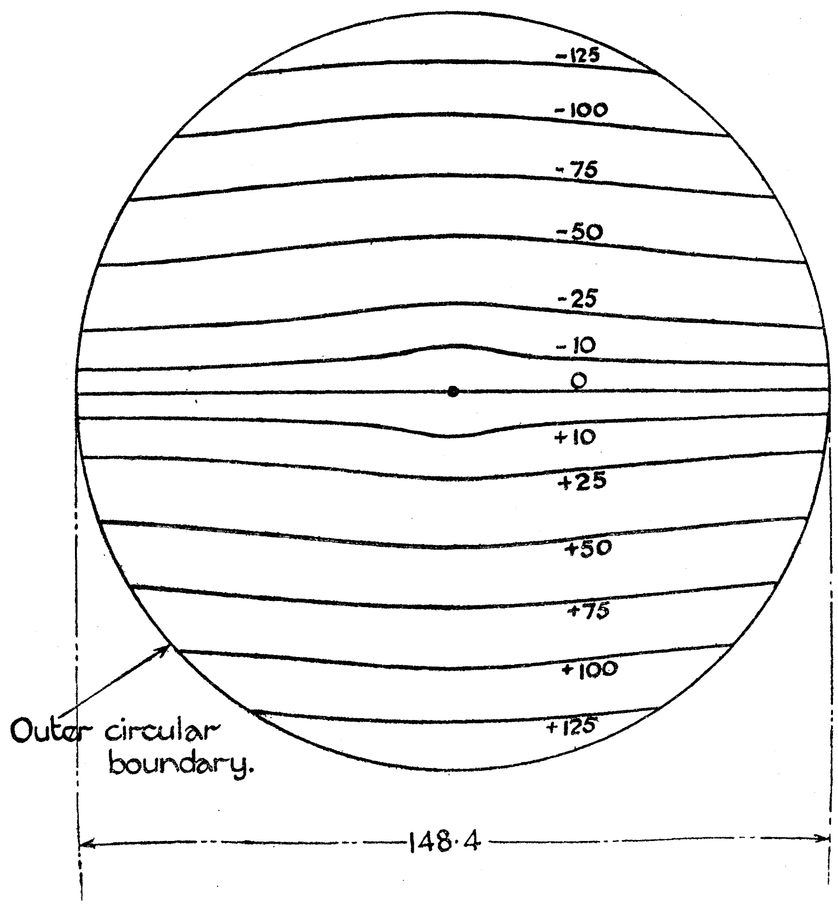


FIG. 2.—Approximate stream-lines are drawn for flow of the type described in fig. 1. The radius R of the outer boundary is here increased to 148·4. The fixed inner cylinder, of unit radius, is represented by the black circle. Curves are drawn for constant values of ψ/U (*see* equation 19 of text). Maximum discrepancies of 0·2 U are allowed over both boundaries.

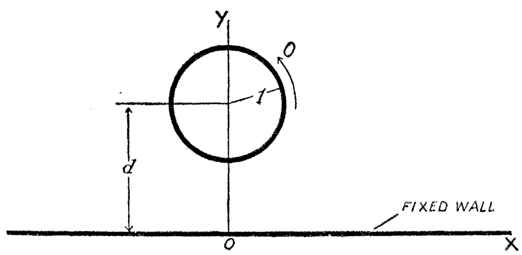
tion is to allow a circular cylinder to translate in the vicinity of an infinite plane boundary or, alternatively, to allow the cylinder to rotate under like conditions. The paragraph which follows show that these types admit of ready solution by the functional method. Both problems, however, are particular cases of the more elaborate system where two

* *Loc. cit.*

mutually external circular cylinders rotate with specified angular velocities. For this general problem a special treatment will be developed.

§ 3. *Rotating Cylinder in Presence of Fixed Wall.*

Suppose a cylinder of unit radius to rotate with uniform counter-clockwise angular velocity O in the presence of a fixed rigid wall OX. Rectangular axes OX, OY are selected as below, the ordinate of the centre of the cylinder being denoted by d. Write $id \equiv c$ and $A^2 \equiv d^2 - 1$.



Since the motion is unsymmetrical about the axis OX, the stream function is here taken in the general form

$$i\psi = \lambda F_1(\mu) - \mu F_2(\lambda) + f_1(\lambda) - f_2(\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

The boundary conditions may be written

$$2\frac{\partial\psi}{\partial\lambda} = \frac{O}{(\lambda - c)} \quad \text{and} \quad 2\frac{\partial\psi}{\partial\mu} = \frac{O}{(\mu + c)} \quad \text{for} \quad (\lambda - c)(\mu + c) = 1,$$

$$\frac{\partial\psi}{\partial\lambda} = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\mu} = 0 \quad \text{for} \quad \lambda - \mu = 0,$$

and the functional relations equivalent to (15) become

$$\frac{Oi}{2(t-c)} = F_1\left(\frac{-tc - A^2}{t - c}\right) + \left(\frac{tc + A^2}{t - c}\right) F_2'(t) + f_1'(t), \quad . \quad . \qquad (21A)$$

$$-\frac{Oi}{2(t+c)} = F_2\left(\frac{tc - A^2}{t + c}\right) - \left(\frac{tc - A^2}{t + c}\right) F_1'(t) + f_2'(t), \quad . \quad . \quad . \qquad (21B)$$

$$0 = F_1(t) - tF_2'(t) + f_1'(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21C)$$

$$0 = F_2(t) - tF_1'(t) + f_2'(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21D)$$

On elimination of f_1' and f_2' we obtain

$$\frac{Oi}{2(t-c)} = \mathrm{F}_1\left(\frac{-tc-\mathrm{A}^2}{t-c}\right) - \mathrm{F}_1(t) + \left(\frac{t^2+\mathrm{A}^2}{t-c}\right)\mathrm{F}_2'(t), \quad \ldots \quad (22\text{A})$$

$$-\frac{Oi}{2(t+c)} = \mathrm{F}_2\left(\frac{tc-\mathrm{A}^2}{t+c}\right) - \mathrm{F}_2(t) + \left(\frac{t^2+\mathrm{A}^2}{t+c}\right)\mathrm{F}_1'(t). \quad \ldots \quad (22\text{B})$$

Assume

$$\mathrm{F}_1(t) = \frac{\alpha_1 t + \beta_1}{t^2+\mathrm{A}^2}, \quad \mathrm{F}_2(t) = \frac{\alpha_2 t + \beta_2}{t^2+\mathrm{A}^2},$$

which, clearly, determine a single-valued expression for pressure

Substitution in (22) leads to

$$\alpha_1 = -\alpha_2 = -\frac{Oi}{2},$$

$$\beta_1 = +\beta_2 = -\frac{Oci}{2}.$$

Whence, on integration of (21) for f_1 and f_2, the stream function becomes

$$\frac{\psi}{O} = \frac{(\lambda-\mu)}{2}\left[\frac{\lambda-id}{\lambda^2+\mathrm{A}^2} - \frac{\mu+id}{\mu^2+\mathrm{A}^2}\right] + \frac{id}{\mathrm{A}}\left[\tan^{-1}\left(\frac{\lambda}{\mathrm{A}}\right) - \tan^{-1}\left(\frac{\mu}{\mathrm{A}}\right)\right]. \quad . \quad (23)$$

At infinite distances from the origin the stream velocity vanishes. The solution is examined further in § 9.

§ 4. *Application of the Principle of Successive Reflections.*

The foregoing methods have the disadvantage of being tentative. For more general problems the principle of successive reflections, as laid down by STOKES,* will be employed. Applications have been made by LADENBURG† and others, in connection with the motion of spheres. Initially, the general case will be investigated, in which definite boundary conditions, not necessarily purely rotational, are specified over the peripheries of two mutually external cylinders A and B. For the moment, the fluid will be assumed at rest at infinity; although this supposition will be revised at a later stage of the analysis.

In the light of STOKES' method the stream function is expressible in the form

$$\psi = \psi_1 + \psi_{12} + \psi_{121} + \psi_{1212} + \text{etc.} \quad \ldots \ldots \quad (24)$$

The first term ψ_1 is the stream function giving the correct boundary conditions over A and at infinity, on the assumption that B is removed. At each point of the contour

* *Loc. cit.*

† 'Ann. der Physik,' vol. 23, pp. 447-458 (1907).

B certain velocity components are induced due to ψ_1: the *reversed* components will be described as due to ψ_1, and "reflected" from B. The term ψ_{12} is obtained on the hypothesis that A is removed and B reinstated, that its derivatives furnish the above reflected velocities at each point of B, and that the fluid is at rest at infinity. The process is repeated indefinitely.

It may be useful to introduce the more general theory by two simple examples. The results will be applied, although in a different form, in the sequel.

Consider the following pair of stream functions

$$\psi_1 = (\lambda - \mu)\left[f_1(\lambda) - f_2(\mu)\right] + \phi_1(\lambda) + \phi_2(\mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25\text{A})$$

$$\psi_2 = (\lambda - \mu)\left[f_2(\lambda) + \phi'_2(\lambda) - f_1(\mu) - \phi'_1(\mu)\right] - \phi_2(\lambda) - \phi_1(\mu). \quad . \quad (25\text{B})$$

Over the plane $y = 0$ (*i.e.*, $\lambda = \mu$) the values of $\partial\psi_1/\partial\lambda$ and $\partial\psi_1/\partial\mu$ are equal, but of opposite sign, to the corresponding derivatives due to (25B). Further, if the singularities of (25A) all lie on one side of $y = 0$, then those of (25B) lie on the opposite side. For convenience, two functions related in the above manner will be described as "conjugate" with regard to the plane $y = 0$.

A pair which exhibit a similar property with respect to the circular boundary $\lambda\mu = 1$, are

$$\psi_1 = (\lambda\mu - 1)\left[f_1(\lambda) + f_2(\mu)\right] + \phi_1(\lambda) + \phi_2(\mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26\text{A})$$

$$\psi_2 = (\lambda\mu - 1)\left[-f_2\left(\frac{1}{\lambda}\right) - \frac{1}{\lambda}\phi'_2\left(\frac{1}{\lambda}\right) - f_1\left(\frac{1}{\mu}\right) - \frac{1}{\mu}\phi'_1\left(\frac{1}{\mu}\right)\right] - \phi_2\left(\frac{1}{\lambda}\right) - \phi_1\left(\frac{1}{\mu}\right). \quad (26\text{B})$$

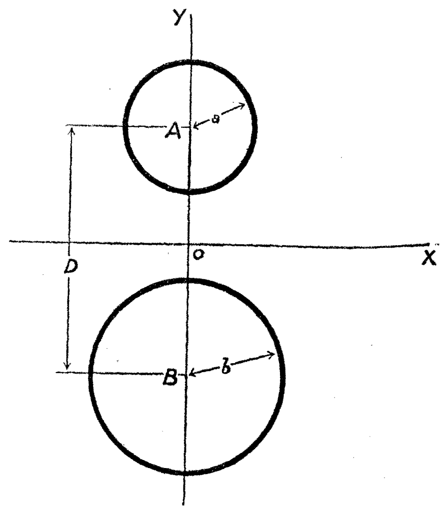
The symbol $\phi_2'(1/\lambda)$ here implies differentiation of $\phi_2(t)$ with respect to t, and subsequent substitution of $1/\lambda$ for t.

These relations provide a basis for the solution of the particular problems in view. Suppose, for example, a cylinder A ($\overline{\lambda - ia}\;\overline{\mu + ia} = 1$) to rotate with unit angular velocity in the presence of a further fixed cylinder B ($\lambda\mu = 1$). The initial stream function ψ_1 of the sequence is obtained on the assumption that B is removed; it is clearly of form

$$2\psi_1 = \log(\lambda - ia) + \log(\mu + ia). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The second function ψ_{12} is determined with A removed and B reinstated. Its singularities must lie within B, and its derivatives over this contour must be equal to the reversed derivatives due to ψ_1. The appropriate function is, clearly, the conjugate of (27), and is immediately obtained by substitution of the relevant functional forms in (26). A continuation of the process would eventually furnish the series (24), and it was by such methods that the general problem was first approached. However, the results indicated a more convenient treatment, to which attention will now be drawn.

§ 5.—*On Functions Conjugate with regard to a Pair of Cylinders A and B.*



Use will be made of the following transformation from the variables x, y to ρ, σ, or, alternatively, from λ, μ to Ω, ω.

$$\lambda \equiv x + iy = -i\gamma \coth\left(\frac{\rho + i\sigma}{2} + \delta\right) \equiv -i\gamma \coth(\Omega + \delta)$$

$$\mu \equiv x - iy = +i\gamma \coth\left(\frac{\rho - i\sigma}{2} + \delta\right) \equiv +i\gamma \coth(\omega + \delta). \quad . \quad . \qquad (28)$$

The constants γ and δ are selected so that the two mutually external circles A and B correspond to $\rho = -K$ and $\rho = +K$, respectively. If a, b denote the radii, and D the distance separating the centres, then

$$\gamma = a \sinh(K - 2\delta) = b \sinh(K + 2\delta), \quad . \quad . \quad . \quad . \qquad (29\text{A})$$

$$\gamma D = ab \sinh 2K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29\text{B})$$

It is assumed throughout that K is positive and $\geq 2\delta$. Values of $\rho < -K$ lead to points lying within A; and values of $\rho > +K$ determine points within B.

For further convenience, write

$$K - 2\delta = \kappa_1 \quad \text{and} \quad K + 2\delta = \kappa_2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

and let

$$\mathfrak{A} \sinh \kappa_2 \equiv (\lambda - i\gamma \coth \kappa_1)(\mu + i\gamma \coth \kappa_1) - a^2, \quad . \quad . \quad . \quad . \qquad (31)$$

$$\mathfrak{B} \sinh \kappa_1 \equiv (\lambda + i\gamma \coth \kappa_2)(\mu - i\gamma \coth \kappa_2) - b^2. \quad . \quad . \quad . \quad . \qquad (32)$$

Then, alternatively, in terms of the variables Ω, ω

$$\mathfrak{A} = \frac{\gamma^2 \sinh(\omega + \Omega + K)}{\sinh \kappa_1 \sinh \kappa_2 \sinh(\omega + \delta) \sinh(\Omega + \delta)}, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\mathfrak{B} = -\frac{\gamma^2 \sinh(\omega + \Omega - K)}{\sinh \kappa_1 \sinh \kappa_2 \sinh(\omega + \delta) \sinh(\Omega + \delta)}. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

It will now be shown that the following pair of stream functions ψ_n and ψ_{n+1} are conjugate with respect to the circle B, provided the functions g_n, g_{n+1}, g_{n+2} are suitably related.

$$\psi_n = -\mathfrak{A} g_{n+1}(\Omega + mK) + \mathfrak{B} g_n(\Omega + mK). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\psi_{n+1} = -\mathfrak{B} g_{n+2}(-\omega + \overline{m+1}\,K) + \mathfrak{A} g_{n+1}(-\omega + \overline{m+1}\,K) \quad . \quad . \quad (36)$$

The conditions necessary are that the composite function

$$\Psi \equiv \psi_n + \psi_{n+1} = -\mathfrak{A}\,[g_{n+1}(\Omega + mK) - g_{n+1}(-\omega + \overline{m+1}\,K)]$$
$$- \mathfrak{B}\,[g_{n+2}(-\omega + \overline{m+1}\,K) - g_n(\Omega + mK)] \quad . \quad . \quad (37)$$

shall have derivatives $\partial\Psi/\partial\Omega$, $\partial\Psi/\partial\omega$, which *vanish* over the circle B, *i.e.*, when $\mathfrak{B} = 0$.

Over this boundary $\Omega + \omega = K$, and so, identically,

$$g_{n+1}(\Omega + mK) - g_{n+1}(-\omega + \overline{m+1}\,K) = 0.$$

Hence the only terms which remain on differentiation for Ω are

$$\left(\frac{\partial\Psi}{\partial\Omega}\right)_B = -(\mathfrak{A})_B \frac{\partial}{\partial\Omega}[g_{n+1}(\Omega + mK)] - \left(\frac{\partial\mathfrak{B}}{\partial\Omega}\right)_B [g_{n+2}(-\omega + \overline{m+1}\,K) - g_n(\Omega + mK)].$$

On application of (33) and (34), and further reduction, it follows that $\left(\frac{\partial\Psi}{\partial\Omega}\right)_B = 0$, provided

$$g_{n+2}(\Omega + mK) - g_n(\Omega + mK) = \sinh 2K \frac{\partial}{\partial\Omega}[g_{n+1}(\Omega + mK)].$$

A similar equation determines the condition that $\left(\frac{\partial\Psi}{\partial\omega}\right)_B = 0$. Hence, the fundamental relation which ensures the conjugacy of the stream functions is

$$g_{n+2}(t) - g_n(t) = \sinh 2K \frac{\partial}{\partial t}[g_{n+1}(t)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Another pair, which are conjugate with respect to B for the same condition, may be obtained by interchange of the variables Ω and ω in (35) and (36).

It may, further, be established that (38) also represents the condition for which the following are conjugate *with respect to the cylinder A*.

$$\psi_n = -\mathfrak{B} g_{n+1}(-\Omega + m\,\mathrm{K}) + \mathfrak{A} g_n(-\Omega + m\,\mathrm{K}). \quad . \quad . \quad . \quad . \qquad (39)$$

$$\psi_{n+1} = -\mathfrak{A} g_{n+2}(\omega + \overline{m+1}\,\mathrm{K}) + \mathfrak{B} g_{n+1}(\omega + \overline{m+1}\,\mathrm{K}) \quad . \quad . \quad . \qquad (40)$$

A second pair is again obtained by interchange of Ω and ω.

Inspection of (35) and (36) indicates that, *in general*, if the singularities of ψ_n all lie within A, then those of ψ_{n+1}, all lie within B : and parallel conclusions may be drawn with regard to (39) and (40). In the application to any special case, however, an examination of the relevant singularities becomes desirable. The conditions at infinity, moreover, need careful inspection : since, even if the derivatives of ψ_n vanish at infinity, the same property does not necessarily attach to ψ_{n+1}. This difficulty will be illustrated at a later stage.

§ 6. *Application to General Velocity Distributions over the Boundaries.*

The broad application of the foregoing theorems will now be considered, and the details left for treatment in relation to the case of rotation. It will be supposed at the outset that B is at rest ; the extension to more elaborate cases is illustrated subsequently. With B removed, the initial (real) stream function due to A is accepted in a general form

$$\psi_0 = -\mathfrak{A}[g_1(\omega) + g_1(\Omega)] + \mathfrak{B}[g_0(\omega) + g_0(\Omega)]. \quad . \quad . \quad . \quad . \qquad (41)$$

In the manner illustrated on p. 104, successive reflections determine a sequence of conjugate stream functions, which may in the light of § 5 be written :—

$$\psi_1 = -\mathfrak{B}[g_2(-\Omega + \mathrm{K}) + g_2(-\omega + \mathrm{K})] + \mathfrak{A}[g_1(-\Omega + \mathrm{K}) + g_1(-\Omega + \mathrm{K})],$$
$$\psi_2 = -\mathfrak{A}[g_3(\omega + 2\mathrm{K}) + g_3(\Omega + 2\mathrm{K})] + \mathfrak{B}[g_2(\omega + 2\mathrm{K}) + g_2(\Omega + 2\mathrm{K})],$$
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \text{, etc.,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

where

$$g_{n+2}(t) = \sinh 2\mathrm{K} g'_{n+1}(t) + g_n(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

Hence, provided the necessary conditions for convergence and re-arrangement are satisfied, the final stream function becomes

$$\Psi = -\mathfrak{A}\left[\begin{array}{c} g_1(\omega) + g_3(\omega + 2\mathrm{K}) + g_5(\omega + 4\mathrm{K}) + \ldots \\ -g_1(-\omega + \mathrm{K}) - g_3(-\omega + 3\mathrm{K}) - \ldots \\ \text{plus similar terms in } \Omega \end{array}\right]$$
$$+\mathfrak{B}\left[\begin{array}{c} g_0(\omega) + g_2(\omega + 2\mathrm{K}) + g_4(\omega + 4\mathrm{K}) + \ldots \\ -g_2(-\omega + \mathrm{K}) - g_4(-\omega + 3\mathrm{K}) - \ldots \\ \text{plus similar terms in } \Omega \end{array}\right]. \quad . \qquad (44)$$

Each of the functions g_n may clearly be expressed, on application of the recurrence formula (43), in terms of g_1, g_0, and their derivatives. Functions of this class have been studied in detail by SONINE* and others; only the following properties need be noted here.

Suppose $\vartheta_n(a)$ to denote the coefficient of y^n in the expansion (supposed convergent) of $\frac{1}{(1-ay-y^2)}$ in positive powers of y. Thus

$$\vartheta_0(a) + \vartheta_1(a)\,y + \vartheta_2(a)\,y^2 + \ldots = \frac{1}{1-ay-y^2}. \qquad (45)$$

Then

$$\vartheta_0(a) = 1; \quad \vartheta_1(a) = a; \quad \vartheta_2(a) = a^2+1, \text{ etc.,}$$

$$\vartheta_n(a) = a^n + (n-1)\,a^{n-2} + \frac{(n-2)(n-3)}{\underline{|2}}\,a^{n-4} + \ldots \qquad (46)$$

The function $g_{n+2}(t)$ is expressible in terms of $g_0(t)$ and $g_1(t)$ as follows

$$g_{n+2}(t) = \vartheta_{n+1}\left(c\frac{\partial}{\partial t}\right)g_1(t) + \vartheta_n\left(c\frac{\partial}{\partial t}\right)g_0(t). \qquad (47)$$

where $c = \sinh 2\,\mathrm{K}$ and $n \geq 0$.

With $a = 2\sinh\alpha$, equation (46) reduces to the simpler form

$$\vartheta_n(2\sinh\alpha) = \frac{\cosh(n+1)\alpha}{\cosh\alpha} \quad \text{or} \quad \frac{\sinh(n+1)\alpha}{\cosh\alpha}, \qquad (48)$$

according as n is even or odd.

The reduction of (44) for forms of g_1 and g_0, corresponding to cases of rotation, will now be undertaken. A further application, arising from the analogy with elastic theory, suggests itself in relation to problems in which an unloaded flat plate is clamped in a specified manner along a pair of mutually external circular boundaries.† However, the detailed reductions for such cases hardly fall within the range of the paper.

§ 7. *Reduction for a Pair of Rotating Cylinders.*

At the outset the simpler case will be taken in which A is supposed rotating uniformly, and B maintained at rest. Here the initial stream function ψ_0 becomes, on suppression of a constant multiplier

$$\psi_0 = \log(\lambda - i\gamma\coth\kappa_1) + \log(\mu + i\gamma\coth\kappa_1). \qquad (49)$$

* 'Math. Ann.,' vol. 16, pp. 1-9, 71-80 (1880). Further references in 'A Treatise on the Theory of Bessel Functions,' by G. N. WATSON, pp. 353-354 (1922).

† In this case the equation $\nabla^4\psi = 0$ determines the deflection ψ at any point of the plate. For a discussion of the appropriate boundary conditions, *see* 'Phil. Mag.,' vol. 41, p. 599.

Identifying this with the form (41), we obtain, after some reduction and omission of a further constant multiplier, the following expressions for the initial functions of the sequence.

$$g_1(t) = +\sinh(t-\mathrm{K}-\delta)\sinh(t+\delta)\log\left\{\frac{\sinh(t+\mathrm{K}-\delta)}{\sinh(t+\delta)}\right\} \quad . \quad . \qquad (50)$$

$$g_0(t) = -\sinh(t+\mathrm{K}-\delta)\sinh(t+\delta)\log\left\{\frac{\sinh(t+\mathrm{K}-\delta)}{\sinh(t+\delta)}\right\} \quad . \quad . \qquad (51)$$

The reduction of the series which comprise the coefficients of $\mathfrak{A}$ and $\mathfrak{B}$ in (44) may be effected by expansion of (50) and (51) in terms of exponentials, and application of the relations (45)—(48). The analysis is exceedingly laborious, and only the final results will be given.

Let

$$\mathrm{S}_1 = \sum_{r=0}^{\infty}[g_{2r+1}(\omega+2r\mathrm{K}) - g_{2r+1}(-\omega+\overline{2r+1}\,\mathrm{K})] \quad . \quad . \quad . \quad . \qquad (52)$$

$$\mathrm{S}_2 = g_0(\omega) + \sum_{r=0}^{\infty}[g_{2r+2}(\omega+\overline{2r+2}\,\mathrm{K}) - g_{2r+2}(-\omega+\overline{2r+1}\,\mathrm{K})]. \quad . \qquad (53)$$

Then, apart from simple terms, which are reinstated later with others in (60), we obtain

$$\tfrac{1}{2}\mathrm{S}_1 = [\cosh(2\omega-\mathrm{K}) - \cosh\kappa_2]\,\mathrm{E}(\omega) + \sum_{s=2}^{\infty} a_s \sinh s(2\omega-\mathrm{K}) \quad . \quad . \qquad (54)$$

$$\tfrac{1}{2}\mathrm{S}_2 = [\cosh(2\omega+\mathrm{K}) - \cosh\kappa_1]\,\mathrm{E}(\omega) + \sum_{s=2}^{\infty} b_s \sinh s(2\omega+\mathrm{K}) \quad . \quad . \qquad (55)$$

in which

$$\mathrm{E}(\omega) = \log\left[\frac{\sinh(\mathrm{K}\pm\overline{\omega-\delta})\sinh(3\mathrm{K}\pm\overline{\omega-\delta})\sinh(5\mathrm{K}\pm\overline{\omega-\delta})\ldots}{\sinh(\omega+\delta)\sinh(2\mathrm{K}\pm\overline{\omega+\delta})\sinh(4\mathrm{K}\pm\overline{\omega+\delta})\ldots}\right]. \qquad (56)$$

$$a_s = \frac{\sinh^2 2\mathrm{K}}{(\sinh^2 2s\mathrm{K} - s^2\sinh^2 2\mathrm{K})}\left[\frac{2s\cosh\kappa_2\sinh s\kappa_1}{\sinh 2s\mathrm{K}} - \frac{(s-1)\sinh(s+1)\kappa_1}{\sinh(s+1)2\mathrm{K}} - \frac{(s+1)\sinh(s-1)\kappa_1}{\sinh(s-1)2\mathrm{K}}\right] \quad . \qquad (57)$$

$$b_s = \frac{\sinh^2 2\mathrm{K}}{(\sinh^2 2s\mathrm{K} - s^2\sinh^2 2\mathrm{K})}\left[\frac{2s\cosh\kappa_1\sinh s\kappa_2}{\sinh 2s\mathrm{K}} - \frac{(s-1)\sinh(s+1)\kappa_2}{\sinh(s+1)2\mathrm{K}} - \frac{(s+1)\sinh(s-1)\kappa_2}{\sinh(s-1)2\mathrm{K}}\right] \quad . \qquad (58)$$

It is readily established that each of the series so defined is absolutely and uniformly convergent.

The foregoing results develop on the assumption that the cylinder A is maintained in a state of uniform rotation, and that B is at rest. On the reverse supposition, the initial function corresponding to (49), is

$$\chi_0 = \log(\lambda + i\gamma\coth\kappa_2) + \log(\mu - i\gamma\coth\kappa_2). \quad . \quad . \quad . \quad . \qquad (59)$$

equivalent to reversal of the sign of K. The full analysis could now be repeated for the new conditions. It will, however, be sufficiently evident from the reciprocity in the form of the coefficients a_s and b_s, and from the symmetry of the elliptic function E, that the final series would be identical with those already obtained in (54) and (55). The latter are, moreover, clearly applicable to the more general case where *both* cylinders A and B are in a state of steady rotation.

In general, a rigorous treatment of the rotation problem on these lines demands a precise term by term adjustment of the successive stream functions, to ensure the correct conditions at infinity. It is possible to effect this adjustment—in a certain measure—by the addition of simple elements to each such function; but the process becomes unduly complex, and needs no detailed illustration.

A considerably more convenient, if less stringent, procedure, is to proceed with the summation of the functions g in the manner prescribed, and to adjust, as completely as possible, the behaviour of the final result by the inclusion of appropriate terms. A solution conducted on the latter basis has, in point of fact, the advantage that it may cover a somewhat wider problem than would otherwise be embraced. In order to complete the solution for the general case of rotation this second treatment will be adopted.

It appears that all conditions, such as may legitimately be imposed at infinity, can be satisfied by incorporating in the solution a simple "adjusting" function of the following type:

$$\begin{aligned}\Psi' = &- \mathfrak{A}\left[a_1 \sinh(2\omega - K) + A_1\{\cosh(2\omega - K) - \cosh\kappa_2\} + a_0(2\omega - K)\right.\\ &\qquad \left. + \text{ similar terms in } \Omega\right]\\ &+ \mathfrak{B}\left[b_1 \sinh(2\omega + K) + B_1\{\cosh(2\omega + K) - \cosh\kappa_1\} + b_0(2\omega + K)\right.\\ &\qquad \left. + \text{ similar terms in } \Omega\right]\\ &+ \alpha_0(\omega + \Omega). \qquad (60)\end{aligned}$$

For computation it is preferable to express the elliptic function E as a series, and to accept the completed stream function in the modified form

$$\begin{aligned}C\Psi = &- \mathfrak{A}\left[\sum_1^\infty a_s \sinh s(2\omega - K) + A_1\{\cosh(2\omega - K) - \cosh\kappa_2\} + a_0(2\omega - K) + \text{ similar terms in } \Omega\right]\\ &+ \mathfrak{B}\left[\sum_1^\infty b_s \sinh s(2\omega + K) + B_1\{\cosh(2\omega + K) - \cosh\kappa_1\} + b_0(2\omega + K) + \text{ similar terms in } \Omega\right]\\ &+ \frac{2\gamma^2 \sinh 2K}{\sinh\kappa_1 \sinh\kappa_2}\left[\log\left\{\frac{\sinh(K + \omega - \delta)\sinh(K - \omega + \delta)}{\sinh(\omega + \delta)}\right\} + \sum_1^\infty \frac{e^{-3sK}}{s}\frac{\sinh 2s\delta \sinh 2s\omega}{\sinh sK} + \sum_1^\infty \frac{e^{-3sK}}{s}\frac{\cosh 2s\delta\cosh 2s\omega}{\cosh sK} + \text{ similar terms in } \Omega\right]\\ &+ \frac{\alpha_0\gamma^2}{\sinh\kappa_1\sinh\kappa_2}(\omega + \Omega). \qquad (61)\end{aligned}$$

For $s \geq 2$ the constants a_s, b_s are defined by (57) and (58). Of the other eight constants C, a_0, b_0, α_0, a_1, b_1, A_1, B_1, two are redundant; since, as may readily be shown by a simple transformation of (61), the last four of this set occur associated in the pairs $a_1 + b_1$, $A_1 + B_1$. Six effective constants remain for determination by the conditions of the problem, which must now be specified precisely.

§ 8. *Determination of Constants.*

The cylinders A, B are assumed to be rotating with uniform counter-clockwise angular velocities O_a, O_b, respectively. The nature of the stream at infinity will be examined later.

For A, with $\omega + \Omega = -K$, the conditions imposed become, after some reduction,

$$\frac{\partial \Psi}{\partial \omega} = \frac{\partial \Psi}{\partial \Omega} = \frac{\gamma^2 O_a}{2 \sinh (\omega + \delta) \sinh (\Omega + \delta) \sinh \kappa_1}. \quad . \quad . \quad . \quad . \quad (62)$$

Whilst for B, with $\omega + \Omega = +K$,

$$\frac{\partial \Psi}{\partial \omega} = \frac{\partial \Psi}{\partial \Omega} = -\frac{\gamma^2 O_b}{2 \sinh (\omega + \delta) \sinh (\Omega + \delta) \sinh \kappa_2}. \quad . \quad . \quad . \quad (63)$$

A direct verification of the solution may now be conducted, by evaluation of the derivatives of (61) with respect to one or other of the variables; but the reductions are heavy, and will be omitted. The following four relations are found sufficient to assure correct conditions over the cylinders:—

$$CO_a \sinh \kappa_2 = 4 (A_1 + B_1) \cosh \kappa_2 + \alpha_0 \cosh \kappa_1 + 8 a_0 K$$
$$+ 4 b_0 \sinh 2K - 2 \sinh 2K \cosh \kappa_1 + 4 \cosh 2K \sinh \kappa_1, \quad . \quad (64\text{A})$$

$$-CO_b \sinh \kappa_1 = 4 (A_1 + B_1) \cosh \kappa_1 + \alpha_0 \cosh \kappa_2 - 4 a_0 \sinh 2K$$
$$- 8 b_0 K + 2 \sinh 2K \cosh \kappa_2 - 4 \cosh 2K \sinh \kappa_2, \quad . \quad . \quad . \quad (64\text{B})$$

$$0 = 4 (a_1 + b_1) \sinh 2K - 4 (A_1 + B_1) \cosh 2K - \alpha_0 + 2 \sinh 2K - \frac{2 \sinh 2\kappa_1}{\cosh 2K}, \quad (64\text{C})$$

$$0 = -4 (a_1 + b_1) \sinh 2K - 4 (A_1 + B_1) \cosh 2K - \alpha_0 - 2 \sinh 2K + \frac{2 \sinh 2\kappa_2}{\cosh 2K}. \quad (64\text{D})$$

One further relation may be extracted by reference to the behaviour of the function (61) at infinity. For this purpose it is preferable to revert to the original variables λ, μ, and to utilise the formulæ

$$e^{2\omega} = \left(\frac{\mu + i\gamma}{\mu - i\gamma}\right) e^{-2\delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65\text{A})$$

$$e^{2\Omega} = \left(\frac{\lambda - i\gamma}{\lambda + i\gamma}\right) e^{-2\delta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65\text{B})$$

The total derivatives of Ψ at infinity may then be shown reducible to

$$C\left(\frac{\partial\Psi}{\partial\lambda}\right)_\infty = 2\mu\left[\sum_1^\infty\left(a_s\frac{\sinh s\kappa_2}{\sinh\kappa_2}+b_s\frac{\sinh s\kappa_1}{\sinh\kappa_1}\right)+\frac{a_0\kappa_2}{\sinh\kappa_2}+\frac{b_0\kappa_1}{\sinh\kappa_1}\right]$$
$$+2i\gamma\left[\begin{array}{c}(A_1+B_1)+\dfrac{a_0}{\sinh\kappa_2}(\kappa_2\coth\kappa_1-1)-\dfrac{b_0}{\sinh\kappa_1}(\kappa_1\coth\kappa_2-1)\\ -\sum_1^\infty\dfrac{a_s}{\sinh\kappa_2}(s\cosh s\kappa_2-\coth\kappa_1\sinh s\kappa_2)\\ +\sum_1^\infty\dfrac{b_s}{\sinh\kappa_1}(s\cosh s\kappa_1-\coth\kappa_2\sinh s\kappa_1)\end{array}\right], \quad (66)$$

with an equivalent expression for $C(\partial\Psi/\partial\mu)_\infty$ obtained by interchange of λ and μ, and reversal of the sign of i. The equivalent velocity components u_∞, v_∞ are determined by (7), and are *finite* provided

$$\sum_1^\infty\left(a_s\frac{\sinh s\kappa_2}{\sinh\kappa_2}+b_s\frac{\sinh s\kappa_1}{\sinh\kappa_1}\right)+\frac{a_0\kappa_2}{\sinh\kappa_2}+\frac{b_0\kappa_1}{\sinh\kappa_1}=0. \quad . \quad . \quad . \quad . \quad (67)$$

Under these conditions the stream at infinity becomes *constant* in magnitude, and normal to the line of centres, being determined by

$$u_\infty = \frac{4\gamma}{C}\left[\begin{array}{c}(A_1+B_1)+\dfrac{a_0}{\sinh\kappa_2}(\kappa_2\coth\kappa_1-1)-\dfrac{b_0}{\sinh\kappa_1}(\kappa_1\coth\kappa_2-1)\\ -\sum_1^\infty\dfrac{a_s}{\sinh\kappa_2}(s\cosh s\kappa_2-\coth\kappa_1\sinh s\kappa_2)\\ +\sum_1^\infty\dfrac{b_s}{\sinh\kappa_1}(s\cosh s\kappa_1-\coth\kappa_2\sinh s\kappa_1)\end{array}\right], \quad . \quad (68\text{A})$$

$$v_\infty = 0. \quad (68\text{B})$$

It appears that, in general, the value of u_∞ devolves upon the remaining boundary conditions imposed, and cannot be regarded as prescribed.

A final relation ensures that the pressure is single-valued. Reference to (10) shows readily that the only components of (61) which contribute a many-valued expression for the pressure, are included under

$$C\Psi = -(a_0\mathfrak{A} - b_0\mathfrak{B})(\omega+\Omega).$$

Amongst these, the only illegitimate terms* are reducible to the form

$$C\Psi = -\lambda\mu\left(\frac{a_0}{\sinh\kappa_2}-\frac{b_0}{\sinh\kappa_1}\right)(\omega+\Omega).$$

* The terms in $(\omega+\Omega)$ which are retained are equivalent to the element $D_0\,\alpha\sinh\alpha$ in Jeffery's paper, *loc. cit.*, p. 170.

The further requisite condition is, clearly,

$$\frac{a_0}{\sinh \kappa_2} = \frac{b_0}{\sinh \kappa_1}.$$

Equating each ratio to a constant c_0, we have, on reduction of the system (64) and (67), and reversion to the notation of (29), the following relations:—

$$a_1 + b_1 = -\frac{\gamma \tanh 2\mathrm{K}}{\mathrm{D}}. \quad \text{(69A)}$$

$$\frac{\mathrm{C}}{8}(O_a a^2 + O_b b^2) = \gamma^2 + c_0 [\mathrm{K}(a^2 + b^2) + \gamma \mathrm{D}], \quad \text{(69B)}$$

$$-(\mathrm{A}_1+\mathrm{B}_1)\gamma \mathrm{D}(O_a a^2+O_b b^2) = \gamma^2 (O_a a^2 - O_b b^2) + c_0 [2\mathrm{K} a^2 b^2 (O_a - O_b) + \gamma \mathrm{D}(O_a a^2 - O_b b^2)], \quad \text{(69C)}$$

$$\alpha_0 = 2 \sinh(\kappa_2 - \kappa_1) - 4(\mathrm{A}_1 + \mathrm{B}_1) \cosh 2\mathrm{K}, \quad \text{(69D)}$$

$$a_0 = \frac{\gamma c_0}{b} \quad \text{and} \quad b_0 = \frac{\gamma c_0}{a}, \quad \text{(69E)}$$

$$2\mathrm{K} c_0 = \frac{\gamma \tan 2\mathrm{K}}{\mathrm{D}} - \sum_2^\infty \left[a_s \frac{\sinh s\kappa_2}{\sinh \kappa_2} + b_s \frac{\sinh s\kappa_1}{\sinh \kappa_1} \right]. \quad \text{(69F)}$$

The above conditions determine the six effective constants uniquely, subject to the conditions prescribed over the cylinders, and to the imposition of a constant stream at infinity.

We proceed to the consideration of special cases.

§ 9. *Special Cases: Series Terms Absent.*

At the outset, it may be observed that the series terms of (61) disappear, provided $1/\mathrm{C}$ vanishes. In general, this condition is satisfied when

$$O_a a^2 + O_b b^2 = 0. \quad (70)$$

The solution now reduces to a relatively simple form, viz. :—

$$\frac{2 \tanh 2\mathrm{K}}{(O_a a^2 - O_b b^2)} \Psi = \rho + \frac{\sinh(\rho - \mathrm{K})[\cosh(\rho + \mathrm{K}) \cos \sigma - \cosh \kappa_1]}{\cosh 2\mathrm{K}[\cosh(\rho + 2\delta) - \cos \sigma]}. \quad (71)$$

At infinity the velocity components are (*see* (68))

$$u_\infty = -\frac{(O_a a^2 - O_b b^2)}{2\mathrm{D}} \quad \text{and} \quad v_\infty = 0. \quad (72)$$

With $a = b$ (*i.e.*, $\delta = 0$) we have the corresponding conditions $O_a = -O_b$; and the solution becomes

$$\frac{\tanh 2\mathrm{K}}{a^2 O_a} \Psi = \rho + \frac{\sinh(\rho - \mathrm{K})[\cosh(\rho + \mathrm{K}) \cos \sigma - \cosh \mathrm{K}]}{\cosh 2\mathrm{K}[\cosh \rho - \cos \sigma]}, \quad (73)$$

which may be identified with JEFFERY's solution for this case by conversion of the quantities γ, ρ, σ, K, O_a into his equivalent symbols a, $-\alpha$, $-\beta$, α, ω.

A further sub-case of interest is where the radius b becomes infinite. The solution then relates to steady rotation of a circular cylinder A in the presence of a fixed wall,* and becomes, with $K = -2\delta$

$$\frac{\gamma}{a^3 O_a \cosh 2K} \Psi = \rho + \frac{\sinh(\rho - K)[\cosh(\rho + K)\cos\sigma - \cosh 2K]}{\cosh 2K[\cosh(\rho - K) - \cos\sigma]} . \quad (74)$$

Here, the velocity components at infinity *vanish*. In order to compare this with the result already obtained in § 3, the following substitutions should be made in (23)—

$$A = \gamma \quad \text{and} \quad d = \cosh 2K,$$

with

$$\lambda = -i\gamma \coth\left(\frac{\rho - K + i\sigma}{2}\right)$$

$$\mu = +i\gamma \coth\left(\frac{\rho - K - i\sigma}{2}\right). \quad (75)$$

The solutions are readily shown to agree.

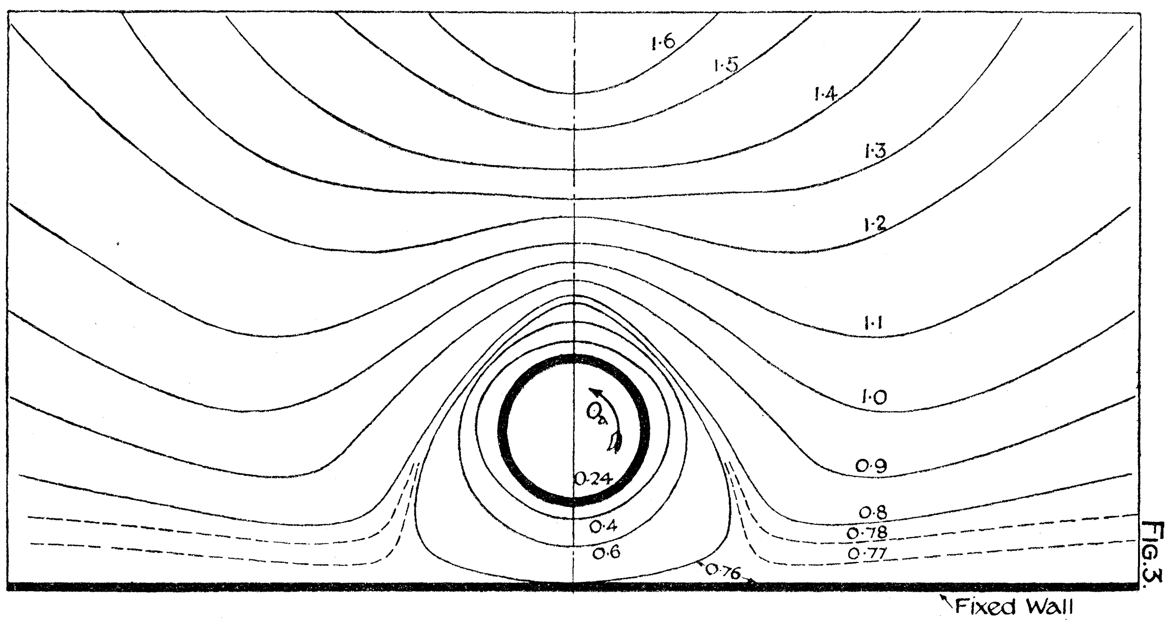


FIG. 3.—The stream-lines shown are due to a cylinder rotating in the presence of a fixed wall. The centre of the cylinder lies at a diameter's distance from the wall, and the fluid is at rest at infinity. Curves are drawn for constant values of Ψ/O_a (*see* equation 74 of text).

Fig. 3 represents the results of calculations for a special case of (74), in which

$$a = 1; \quad \gamma = \sinh 2K; \quad 2 = \cosh 2K.$$

* JEFFERY proceeds to this case from his solution for a cylinder rotating *inside* a non-concentric circular vessel.

The centre of A is selected at a diameter's distance from the plane. It may be noted that the stream-line $\Psi/O_a = 0\cdot760$ is compounded of the fixed plane, and of a closed curve, touching the plane and enclosing eddying fluid.

§ 10. *Special Cases : Stream at Rest at Infinity, and "Planetary" Systems.*

It has already been shown that a unique normal solution to the general problem may be obtained such that the velocity components at infinity are *constant.* In general, however, the precise magnitude of the stream at infinity must be regarded as devolving upon the conditions prescribed for the cylinders. It will, indeed, be readily evident that any additional terms which might be incorporated in the solution in order to bring that stream to rest, would necessarily have to satisfy conditions which have already been shown in § 2 to lead to anomalous functions.

However, an examination of equations (68) and (69) indicates that the fluid can always be brought to rest at infinity by choice of a suitable ratio between the angular velocities O_a and O_b. A relatively simple case is obtained* when

$$a = b \qquad \text{and} \qquad O_a = +O_b. \tag{76}$$

Here $\delta = 0$ and

$$a_s = b_s = -\frac{\sinh K \sinh^2 2K}{2\cosh(s-1)K\cosh sK\cosh(s+1)K[\sinh 2sK + s\sinh 2K]} \tag{77A}$$

for $s \geq 2$,

$$a_1 + b_1 = -\frac{\sinh^2 K}{\cosh 2K}, \tag{77B}$$

$$\frac{CO_a}{4} = \sinh^2 K + c_0(2K + \sinh 2K), \tag{77C}$$

$$A_1 + B_1 = \alpha_0 = 0, \tag{77D}$$

$$a_0 = b_0 = c_0 \sinh K. \tag{77E}$$

The derivatives at infinity now become (*see* (66))

$$C\left(\frac{\partial\Psi}{\partial\lambda}\right)_\infty = 2\mu\left(\frac{-\sinh^2 K}{\cosh 2K} + 2\sum_2^\infty a_s \frac{\sinh sK}{\sinh K} + 2c_0 K\right), \tag{78A}$$

$$C\left(\frac{\partial\Psi}{\partial\mu}\right)_\infty = 2\lambda\left(\frac{-\sinh^2 K}{\cosh 2K} + 2\sum_2^\infty a_s \frac{\sinh sK}{\sinh K} + 2c_0 K\right). \tag{78B}$$

Hence, if

$$2c_0 K = \frac{\sinh^2 K}{\cosh 2K} - 2\sum_2^\infty a_s \frac{\sinh sK}{\sinh K}, \tag{79}$$

we have sufficient to determine the solution for equal cylinders rotating with equal angular velocities *in the same sense.* Under these conditions the stream is undisturbed at infinity.

* For the corresponding case $a = b$, and $O_a = -O_b$ *see* equation (73).

The expressions (78) illustrate a somewhat more general class of problem. Under conditions where (79) is *not satisfied*, the derivatives become of form

$$C\left(\frac{\partial \Psi}{\partial \lambda}\right)_\infty = 2\mu H, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80\text{A})$$

$$C\left(\frac{\partial \Psi}{\partial \mu}\right)_\infty = 2\lambda H, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80\text{B})$$

where H is now regarded as some specified constant. At infinite distances the fluid now circulates uniformly about the origin with angular velocity 4H/C; and the funct on Ψ may here be interpreted as determining the stream-lines, relative to the selected co-ordinate axes, when the cylinders are rotating about their own axes *and in addition about the origin* with uniform angular velocity 4H/C.

More generally, the relation (66) indicates that a possible type of steady motion in a fluid at rest at infinity, is where any pair of rotating cylinders are rotated appropriately as a planetary system about a particular "focus" on their line of centres. It will be sufficient to examine a few special cases.

Suppose $a = b$, with O_a and O_b arbitrary. Here, with a_s defined by (77A), we have

$$C\left(\frac{\partial \Psi}{\partial \lambda}\right)_\infty = 2\mu\left[2c_0 K - \frac{\sinh^2 K}{\cosh 2K} - 2\sum_2^\infty a_s \frac{\sinh sK}{\sinh K}\right] + 2i\gamma\,(A_1 + B_1), \quad . \quad (81)$$

together with the relations (69A–E), which become

$$a_1 + b_1 = -\frac{\sinh^2 K}{\cosh 2K}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82\text{A})$$

$$\frac{C}{8}a^2(O_a + O_b) = -(A_1 + B_1)\frac{(O_a + O_b)}{(O_a - O_b)}\gamma D = \gamma^2 + c_0(2Ka^2 + \gamma D), \quad . \quad (82\text{B})$$

$$a_0 = b_0 = \frac{c_0\gamma}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82\text{C})$$

$$\alpha_0 = -4(A_1 + B_1)\cosh 2K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82\text{D})$$

If Y denote the ordinate of the "focus" of revolution, and O_c the corresponding angular velocity of the system about that point, we have

$$CO_c = 4\left[2c_0 K - \frac{\sinh^2 K}{\cosh 2K} - 2\sum_2^\infty a_s \frac{\sinh sK}{\sinh K}\right], \quad . \quad . \quad . \quad . \quad . \quad (83)$$

$$CO_c Y = 4\gamma\,(A_1 + B_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Hence, from (82D)

$$Y = -\frac{a^2}{2D}\left(\frac{O_a - O_b}{O_c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

The series terms in Ψ' may be shown to disappear if

$$a^2(O_a + O_b) = (2Ka^2 + \gamma D)\frac{O_c}{K}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

a relation which provides the simplest illustration of the type required. Under these conditions the stream function reduces to

$$\frac{2D\sinh^2 K}{\gamma}\Psi' = \frac{Ka^3(O_a - O_b)\sinh(\rho - K)[\cosh(\rho + K)\cos\sigma - \cosh K] - 2O_c\gamma^2 D(\rho\sinh\rho\cosh K - K\cosh\rho\sinh K)}{Ka(\cosh\rho - \cos\sigma)}$$

$$+ a^2\cosh 2K(O_a - O_b)\rho, \quad . \quad . \quad (87)$$

where

$$O_c = \frac{a^2 K(O_a + O_b)}{(2Ka^2 + \gamma D)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

and

$$Y = -\frac{(2Ka^2 + \gamma^2)}{2DK}\left(\frac{O_a - O_b}{O_a + O_b}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

With $O_a = O_b$ we have the simple expression

$$-\frac{Ka\sinh^2 K}{\gamma^3 O_c}\Psi' = \frac{(\rho\sinh\rho\cosh K - K\cosh\rho\sinh K)}{(\cosh\rho - \cos\sigma)}, \quad . \quad . \quad . \quad (90)$$

where

$$O_c = \frac{O_a}{\left(1 + \frac{\sinh 2K}{2K}\right)} \quad \text{and} \quad Y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

In Cartesian co-ordinates this becomes

$$\frac{2}{O_c}\Psi' = (x^2 + y^2) - 2\gamma\frac{\coth K}{K}y\coth^{-1}\left(\frac{x^2 + y^2 + \gamma^2}{2y\gamma}\right). \quad . \quad . \quad . \quad (92)$$

Typical stream-lines are shown plotted in fig. 4 for $a = 1$ and $D = 4$. Here

$$\gamma = \sqrt{3}; \quad \coth K = 1{\cdot}154; \quad K = 1{\cdot}317; \quad \text{and} \quad \frac{O_a}{O_c} = 3{\cdot}63.$$

A notable feature is the "figure of eight" stream-line $\Psi'/O_c = 1\cdot 5$, which indicates a stagnation point at the origin (*i.e.*, the central "focus" of revolution).

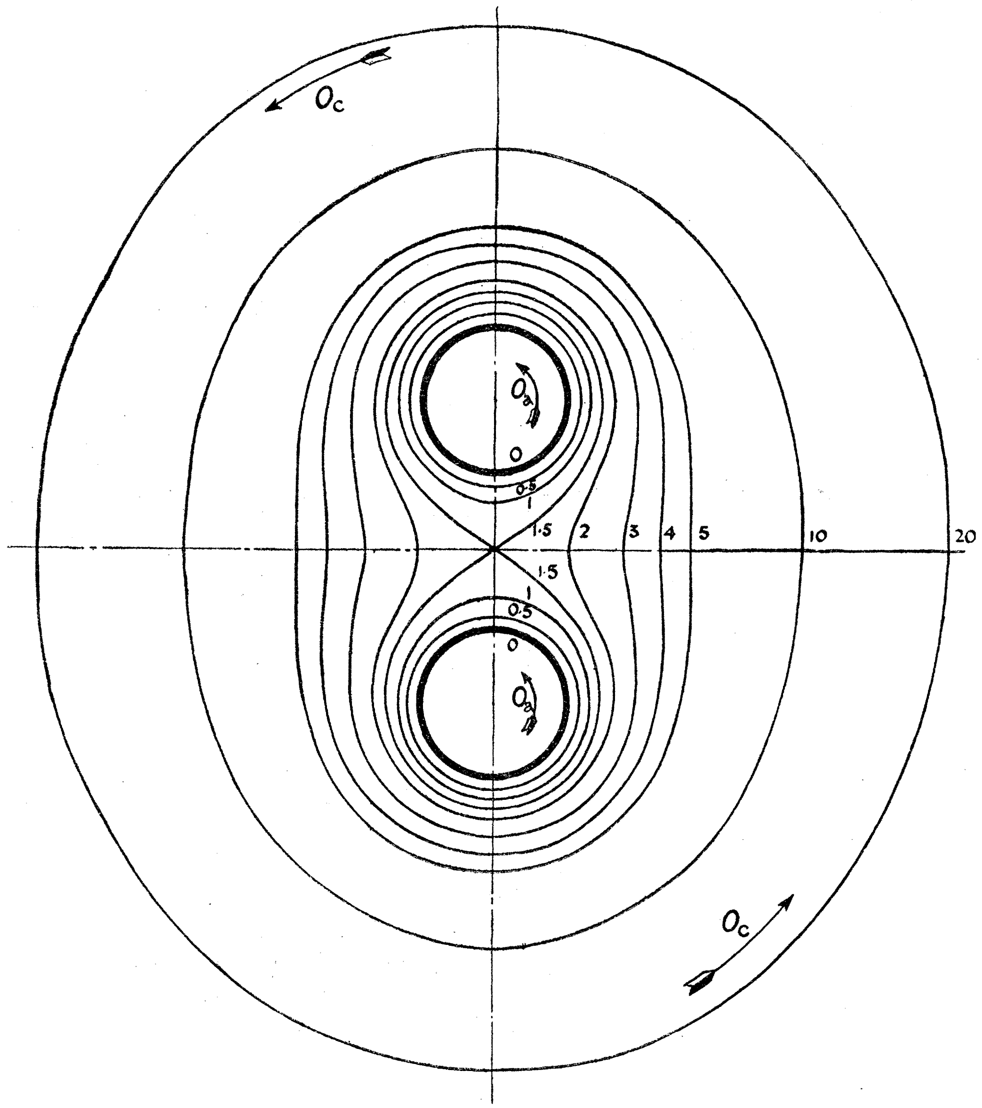


FIG. 4.—The stream-lines represent the flow for a system in which two equal circular cylinders rotate with equal angular velocities O_a. The fluid at infinity circulates, relative to the axes shown with constant angular velocity O_c about the origin, where $O_a/O_c = 3 \cdot 63$. Curves are drawn for constant values of Ψ/O_c (*see* equation 92 of text).

§ 11. *Special Cases: Cylinders in Contact.*

It may be of some interest to examine the form of the general solution when the cylinders A and B are *brought into contact.*

The degenerate form of (61), equivalent to this case, is obtained by a substitution of type

$$K = \varepsilon K' ; \quad \delta = \varepsilon \delta' ; \quad \Omega = \varepsilon \Omega' ; \quad \omega = \varepsilon \omega' ; \quad \gamma = a\varepsilon\kappa'_1 = b\varepsilon\kappa_2' ;$$

in which ε becomes indefinitely small.

The equivalent transformation of variables is

$$\frac{\rho' + i\sigma'}{2} + \delta' = -\frac{ia\kappa_1'}{\lambda}, \quad \text{(93A)}$$

$$\frac{\rho' - i\sigma'}{2} + \delta' = +\frac{ia\kappa_1'}{\mu}. \quad \text{(93B)}$$

A consideration of the formulæ (69) will readily indicate that the constants a_0, b_0, C, $\varepsilon(A_1 + B_1)$, and $\varepsilon\alpha_0$ are all finite and definite, and that $a_s + b_s = 0$ ($s \geq 1$). The solution now reduces to an expression of form

$$C\Psi = -\frac{4ab(\rho' + K')}{[(\rho' + \delta')^2 + \sigma'^2]}[\varepsilon(A_1 + B_1)\{(\rho' - K')^2 - \sigma'^2 - \kappa_2'^2\} + 2a_0(\rho' - K')]$$
$$-\frac{8ab(\rho' - K')b_0(\rho' + K')}{[(\rho' + \delta')^2 + \sigma'^2]} + \varepsilon\alpha_0 ab\rho', \quad \text{(94)}$$

equivalent, in Cartesian co-ordinates, to

$$\Psi = \frac{Ay^2}{(x^2 + y^2)} + C(x^2 + y^2) + Fy + \frac{Gy^3}{(x^2 + y^2)^2}. \quad \text{(95)}$$

The constants A, C, F, G must be selected to satisfy the correct conditions over the two cylinders, viz. :—

$$\frac{\partial\Psi}{\partial x} = O_a x \quad \text{and} \quad \frac{\partial\Psi}{\partial y} = O_a(y - a) \quad \text{for} \quad x^2 + y^2 = 2ay, \quad \text{(96A)}$$

$$\frac{\partial\Psi}{\partial x} = O_b x \quad \text{and} \quad \frac{\partial\Psi}{\partial y} = O_b(y + b) \quad \text{for} \quad x^2 + y^2 = -2by. \quad \text{(96B)}$$

The following equations determine the constants :—

$$O_a = 2C - \frac{A}{2a^2} - \frac{G}{2a^3}, \quad \text{(97A)}$$

$$O_b = 2C - \frac{A}{2b^2} + \frac{G}{2b^3}, \quad \text{(97B)}$$

$$-aO_a = \frac{A}{a} + F + \frac{3G}{4a^2}, \quad \text{(97C)}$$

$$+bO_b = -\frac{A}{b} + F + \frac{3G}{4b^2}. \quad \text{(97D)}$$

From the physical standpoint the most rational case is where the peripheral velocities V of the cylinders are equal, *i.e.*

$$aO_a = -bO_b = V. \quad \text{(98)}$$

We find here

$$\frac{(a+b)^2}{V}\Psi = -\frac{6ab\,(a-b)\,y^2}{(x^2+y^2)} - \frac{8a^2b^2y^3}{(x^2+y^2)^2} + \frac{(a-b)}{2}(x^2+y^2) - (a^2+b^2-4ab)\,y. \quad (99)$$

The condition of steady motion is, thus, of the type illustrated in §10, such that the rotating touching cylinders are themselves revolving as a system about a certain focus. It is useless to delay over this result in its general form. In the particular case where $a = b$, the swirl at infinity vanishes, and the solution reduces to

$$\frac{2}{aO_a}\Psi = -\frac{4a^2y^3}{(x^2+y^2)} + y. \quad (100)$$

The fluid moves with constant speed $\frac{1}{2}aO_a$ at infinity. A number of typical stream-lines, corresponding to the case $a = 1$, are represented in fig. 5.

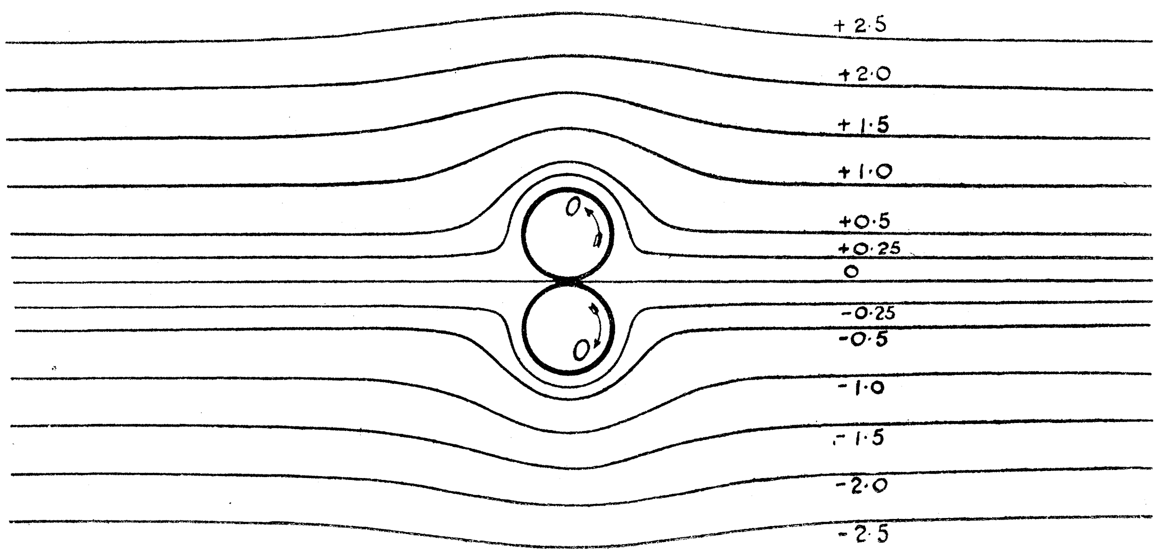


FIG. 5.—The stream-lines represent the flow due to two equal rotating and touching cylinders. The fluid moves with constant speed $aO_a/2$ at infinity. Curves are drawn for constant values of Ψ/O_a (*see* equation 100 of text).

With b infinite in (99) we obtain the stream-lines (relative to the selected co-ordinate axes) due to a cylinder, which rolls *without slip* on a fixed plane, viz.—

$$\frac{\Psi}{aO_a} = \frac{6ay^2}{(x^2+y^2)} - \frac{8a^2y^3}{(x^2+y^2)^2} - y. \quad (101)$$

For comparison we set down the equivalent result for a cylinder rotating, and slipping with relative velocity $2aO_a$, against a fixed plane.

$$\frac{\Psi}{aO_a} = -\frac{2ay^2}{(x^2+y^2)} + y. \quad (102)$$

It is of some importance to observe here that at the point of contact the first derivatives of (101) and (102) behave normally; whereas in both cases the higher derivatives become infinite. It will be sufficient to examine (102). At any point, the velocity components in polar co-ordinates become

$$\frac{u}{aO_a} = \frac{4a}{r}(\sin\theta - \sin^3\theta) - 1, \quad \ldots \quad (103)$$

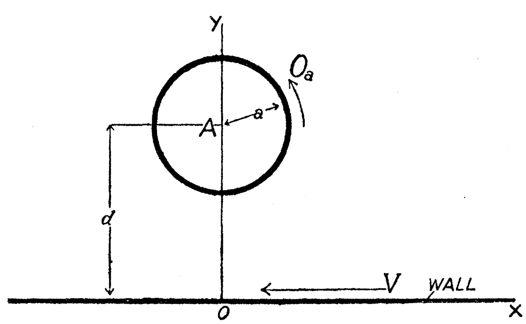
$$\frac{v}{aO_a} = \frac{4a}{r}\cos\theta\sin^2\theta. \quad \ldots \quad (104)$$

A point r, θ, near the origin, and lying within the fluid, must be such that $r < 2a\theta$; whereas on the plane $(y = 0)$ we have $\theta = 0$, and on the cylinder $r = 2a\theta$. Hence, at the origin, the first derivatives remain *finite*, and determine velocity components consistent with the imposed conditions. On the other hand, the derivative $\partial^2\Psi/\partial y^2$ at the origin is readily seen to be infinite, as might be expected from the discontinuity of velocity imposed there. In the case of (101), although no relative motion occurs at the point of contact, this derivative also becomes infinite.*

Such questions are, perhaps, not without some interest in connection with the theory of lubrication, and two further sections are devoted to a fuller discussion of contact problems from a different and, possibly, more illuminating angle.

§ 12. *Special Cases: Cylinder Rotating in Presence of Translating Wall.*

The solution will first be derived for a cylinder A, which both rotates with angular velocity O_a, and translates with speed V, in the presence of a fixed plane boundary.† The stream function obtained must be interpreted as determining the stream-lines relative to the co-ordinate axes, a translational speed —V being assumed imparted to the entire system. The result will then be examined critically as the cylinder is brought into close vicinity to the wall. The following diagram illustrates the particular system under discussion.



When b is very large we have κ_2 small, such that $\gamma = b\kappa_2$, and it readily follows from

* See also *infra*, p. 125.

† This case is also deducible from JEFFERY's results.

(57) and (58) that both $a_s b$ and $b_s b$ remain finite. The conditions of the problem require also that $bO_b = +V$. From (69) we find

$$C = \frac{8c_0 K b}{V},$$

$$b(a_1 + b_1) = -\gamma \tanh 2K,$$

$$(A_1 + B_1) = -c_0 \left[\frac{2Ka^2}{\gamma V} O_a - 1\right],$$

$$\alpha_0 + 2 \sinh K = -4(A_1 + B_1) \cosh 2K,$$

with

$$a_0 = \frac{\gamma c_0}{b} \quad \text{and} \quad b_0 = \frac{\gamma c_0}{a}.$$

It follows that the series terms associated with $\mathfrak{A}$ and $\mathfrak{B}$ in (61) vanish; and a further simple reduction shows that the logarithmic terms and related series differ by a constant from $\omega + \Omega$. The stream function finally simplifies to

$$\frac{2K}{aV}\Psi = S \frac{\sinh(\rho + K)[\cosh(\rho - K)\cos\sigma - 1]}{[\cosh(\rho - K) - \cos\sigma]} - \sinh 2K \frac{(\rho + K)\sinh(\rho - K)}{[\cosh(\rho - K) - \cos\sigma]} + \rho S \cosh 2K, \quad (105)$$

where

$$S = \frac{2KaO_a}{V \sinh 2K} - 1. \quad (106)$$

The transformation of variables is here

$$\lambda = -i\gamma \coth\left(\frac{\rho - K + i\sigma}{2}\right), \quad (107\text{A})$$

$$\mu = +i\gamma \coth\left(\frac{\rho - K - i\sigma}{2}\right), \quad (107\text{B})$$

in which

$$\gamma = a \sinh 2K, \quad (108\text{A})$$

$$d = a \cosh 2K, \quad (108\text{B})$$

where d denotes the distance of the centre of the cylinder from the plane $y = 0$. When $V = 0$ the solution may be identified with (74) of § 9.

The following form, equivalent to (105) in the alternative system of variables λ and μ, is useful in connection with the computation of the stress components :—

$$\frac{4K}{V}\Psi = -\gamma S[(\lambda - id)(\mu + id) - a^2]\left[\frac{1}{\lambda^2 + \gamma^2} + \frac{1}{\mu^2 + \gamma^2}\right]$$
$$- \tfrac{1}{2}[i(\lambda - \mu) - 2Sd] \log \frac{(\mu + i\gamma)(\lambda - i\gamma)}{(\mu - i\gamma)(\lambda + i\gamma)} - 2Ki(\lambda - \mu). \quad (109)$$

Expressions for the mean pressure p at any point, and for the individual stress components may be derived on application of the formulæ (10) and (11). Only the results obtained will be stated.

Let ρ_0 denote the density of the fluid, and ν the coefficient of kinematic viscosity. Then for points on the plane $y = 0$

$$p_{xx} = p_{yy} = -p = \frac{4\gamma S}{K}(\nu\rho_0 V)\frac{xd}{(x^2+\gamma^2)^2}, \quad . \quad . \quad . \quad . \quad (110\text{A})$$

$$p_{xy} = p_{yx} = \frac{2\gamma}{K}(\nu\rho_0 V)\left[\frac{2\gamma^2 S}{(x^2+\gamma^2)^2} - \frac{(S-1)}{(x^2+\gamma^2)}\right]. \quad . \quad . \quad . \quad (110\text{B})$$

Whilst for points on the cylinder

$$\frac{2\gamma^2 Kp}{a(\nu\rho_0 V)} = 2\sin\sigma(S - \sinh^2 2K) - S\cosh 2K \sin 2\sigma, \quad . \quad . \quad . \quad (111\text{A})$$

$$\frac{2K\gamma}{V}\frac{\partial^2\Psi'}{\partial\lambda^2} = [(S+1)\cosh 2K + S\cos\sigma]\frac{\sinh^3\left(K - \frac{i\sigma}{2}\right)}{\sinh\left(K + \frac{i\sigma}{2}\right)}, \quad . \quad . \quad (111\text{B})$$

$$\frac{2K\gamma}{V}\frac{\partial^2\Psi'}{\partial\mu^2} = [(S+1)\cosh 2K + S\cos\sigma]\frac{\sinh^3\left(K + \frac{i\sigma}{2}\right)}{\sinh\left(K - \frac{i\sigma}{2}\right)}. \quad . \quad . \quad (111\text{C})$$

The latter expressions, with (11), are sufficient to determine the stress components over the cylinder. The following further results are obtained by integration over one or other of the boundaries. Let X_0, Y_0, M_0 denote, respectively, the viscous drag, the normal force, and the couple operating on the cylinder. Then

$$X_0 = \frac{2\pi}{K}(\nu\rho_0 V), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112\text{A})$$

$$Y_0 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112\text{B})$$

$$M_0{}^* = \frac{4\pi\nu\rho_0 O_a a^2 d}{\sqrt{d^2 - a^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112\text{C})$$

It appears, therefore, that the drag X_0 and the couple M_0 are, respectively, independent of the rotary and the translational motions of the cylinder.

§ 13. *Special Cases: Film of Contact.*

The foregoing results lead to the conclusion that both the viscous drag and the couple tend to infinite values as the cylinder approaches indefinitely close to the plane. In

* This agrees with JEFFERY's formula, *loc. cit.*, p. 172.

order to meet the physical difficulties associated with *absolute* contact, we presuppose a very thin limiting film of fluid, of thickness δ, separating the boundaries. Under these conditions, we have the approximate expressions

$$\gamma = \sqrt{2a\delta}\,; \qquad K = \sqrt{\frac{\delta}{2a}}; \qquad S = \frac{aO_a}{V}\left(1 - \frac{\delta}{3a}\right) - 1, \quad . \quad . \quad . \quad (113)$$

together with $d - a = \delta$.

(*a*) *Solution for Points not in the Immediate Vicinity of the Origin.*—Here, on introduction of polar co-ordinates, the radius vector r may be assumed $> \sqrt{2a\delta}$, and the logarithmic term in (109) is legitimately expansible in the form

$$\log\frac{(\mu + i\gamma)(\lambda - i\gamma)}{(\mu - i\gamma)(\lambda + i\gamma)} = -4\left(\frac{\sqrt{2a\delta}}{r}\right)\sin\theta + \tfrac{4}{3}\left(\frac{\sqrt{2a\delta}}{r}\right)^3 \sin 3\theta - \text{etc.} \quad . \quad (114)$$

When $\sqrt{2a\delta/r}$ is sufficiently small, the stream function will be found adequately represented by the terms

$$\Psi = 2a\,(aO_a - 2V)\frac{y^2}{(x^2 + y^2)} - 4a^2\,(aO_a - V)\frac{y^3}{(x^2 + y^2)^2} + Vy. \quad . \quad . \quad (115)$$

It may be inferred that the solutions obtained in § 11 are sufficiently representative for points which do not lie in the immediate vicinity of the film of contact.

(*b*) *Solution for Points in Close Proximity to the Origin.*—For points lying within the limiting film, the radius vector r becomes $< \sqrt{2a\delta}$, and the logarithm must now be expanded in a series of positive powers of r. Alternatively, we put $\lambda = \gamma\lambda'$, $\mu = \gamma\mu'$, where λ' and μ' are *finite*, and retain the original form of (109).

The general features of the flow in this neighbourhood are, perhaps, sufficiently illustrated by a consideration of the velocity distribution across the narrowest portion of the film of contact.

On differentiation of the more general solution (109), we readily obtain for the velocity u at any point y on the axis $x = 0$

$$\frac{uK}{V} = \frac{\gamma y}{(\gamma^2 - y^2)} + \frac{2\gamma S y\,(\gamma^2 - yd)}{(\gamma^2 - y^2)^2} - \tfrac{1}{2}\log\left(\frac{\gamma - y}{\gamma + y}\right) - K. \quad . \quad . \quad . \quad (116)$$

Under the conditions (113), with δ small and $y \leq \delta$, this may be written in the approximate form

$$\frac{u}{V} = \frac{(S+1)^2}{S} - 1 - S\left(\frac{S+1}{S} - \frac{y}{\delta}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

Whence, with $y = 0$ and $y = \delta$, we have respectively $u = -V$ and $u = aO_a$; and the velocity clearly remains finite and continuous, following a *parabolic* law across the gap. The vertex of the parabola lies within the limiting film, provided $(S + 1)/S > 0$. In particular, when $V = -aO_a$, the speed reaches a maximum $\frac{3}{2}V$ at the centre of the gap, and falls to the value V at either boundary. This parabolic distribution clearly accounts for the infinite value of the derivative $\partial^2\Psi/\partial y^2$, which was noted in § 11 for the case where no relative motion occurs at the point of contact.

The foregoing investigation prompts the somewhat more general inference that absolute contact between two rotating circular cylinders is primarily precluded by the fact that a parabolic velocity distribution develops across the film of contact as the boundaries approach one another, indicative of an infinite rate of shear. The further consideration, that the physical properties of the fluid may become impaired when the film attains molecular dimensions, hardly falls within the purview of a theoretical paper.

§ 14. *Extension of Solution for Single Translating Cylinder.*

The results enumerated in the foregoing paragraphs have been derived on the hypothesis that the inertia terms associated with the equations of flow are negligible. In their present form, therefore, the solutions admit a very restricted application. It has, however, been pointed out in the *Introduction* that the work is primarily intended to clear the ground for a closer scrutiny of the methods proposed by Cowley and Levy.

For completeness, a résumé of the further procedure is given below, in a form which differs slightly from that laid down by the original authors. The early stages of the reductions for the case of a single translating cylinder are appended, in order to illustrate the nature of the difficulties which are latent in a purely analytical application of the method.

Equations (5) and (6) of § 1 may be replaced by the single equation

$$2\nu\frac{\partial^4\psi}{\partial\lambda^2\,\partial\mu^2} = -i\left(\frac{\partial\psi}{\partial\lambda}\frac{\partial^3\psi}{\partial\lambda\,\partial\mu^2} - \frac{\partial\psi}{\partial\mu}\frac{\partial^3\psi}{\partial\lambda^2\,\partial\mu}\right). \qquad (118)$$

In the particular method proposed the variables λ, μ, ψ are all replaced by non-dimensional quantities, such as those defined below

$$\lambda = L\lambda'; \quad \mu = L\mu'; \quad \psi = UL\psi'.$$

Here L and U are length and velocity factors specifying the class of problem under discussion. Thus L defines the scale of the moving system, and U may be taken as the steady velocity of a prescribed boundary.

However, as an analytical convenience (which is readily seen to be permissible), the original variables λ, μ, will be retained, and only the third substitution effected. The equation of flow becomes

$$2\frac{\partial^4\psi'}{\partial\lambda^2\,\partial\mu^2} = -iC\left[\frac{\partial\psi'}{\partial\lambda}\frac{\partial^3\psi'}{\partial\lambda\,\partial\mu^2} - \frac{\partial\psi'}{\partial\mu}\frac{\partial^3\psi'}{\partial\lambda^2\,\partial\mu}\right], \qquad (119)$$

where

$$C \equiv \frac{UL}{\nu}. \qquad (119\text{A})$$

The function ψ' is now viewed as expansible in positive powers of the parameter C, *e.g.*,

$$\psi' = \frac{\psi}{UL} = 2i\,[\psi_0 + C\psi_1 + C^2\psi_2 + C^3\psi_3 + \ldots]. \qquad (120)$$

The series is substituted in (119), and the coefficient of each power of C equated to zero. A system of subsidiary equations is thus derived, determining successively the functions ψ_0, ψ_1, ψ_2, etc. The first three of these equations are written below, differentiations for λ and μ being indicated, for brevity, in suffix notation.

$$\psi_{0_{\lambda\lambda\mu\mu}} = 0, \qquad (121\text{A})$$

$$\psi_{1_{\lambda\lambda\mu\mu}} = \psi_{0_\lambda}\psi_{0_{\lambda\mu\mu}} - \psi_{0_\mu}\psi_{0_{\lambda\lambda\mu}}, \qquad (121\text{B})$$

$$\psi_{2_{\lambda\lambda\mu\mu}} = \psi_{0_\lambda}\psi_{1_{\lambda\mu\mu}} - \psi_{0_\mu}\psi_{1_{\lambda\lambda\mu}} + \psi_{1_\lambda}\psi_{0_{\lambda\mu\mu}} - \psi_{1_\mu}\psi_{0_{\lambda\lambda\mu}}. \qquad (121\text{C})$$

The initial function ψ_0, equivalent to the "slow-motion" solution, is selected to satisfy the boundary conditions of the problem: whilst the remainder are determined subject to the condition that their first derivatives vanish over all the boundaries. In addition, each function (or, alternatively, the final stream function) must furnish a legitimate single-valued expression for the pressure.

In illustration, the analysis will be carried to the second term for the case of uniform flow past a fixed circular cylinder of radius a. Since the method is such that any boundary conditions satisfied by the initial term remain valid for the extended solution, it will clearly be sufficient to commence with the simple function (19). This approximate expression has already been shown to meet the desired conditions to any degree of accuracy, provided the outer radius R be sufficiently large.

Proceeding on this basis, we choose $L = a$ in (119A), and modify the stream function to:—

$$\frac{\psi}{2iUa} = \psi_0 + \left(\frac{Ua}{\nu}\right)\psi_1, \qquad (122)$$

in which

$$\psi_0 = \frac{(\lambda - \mu)\log(\lambda\mu)}{8a\log R}, \qquad (123)$$

and

$$\psi_{1_{\lambda\lambda\mu\mu}} = \psi_{0_\lambda}\psi_{0_{\lambda\mu\mu}} - \psi_{0_\mu}\psi_{0_{\lambda\lambda\mu}}. \qquad (124)$$

The boundary conditions for ψ_1 are $u = v = 0$, both for $\lambda\,\mu = a^2$ and $\lambda\,\mu = R^2$. After suitable reduction (124) becomes

$$\psi_{1_{\lambda\lambda\mu\mu}} = -\frac{(\lambda^2 - \mu^2)}{64a^2(\log R)^2}\left[\frac{\log(\lambda\mu)}{\lambda^2\mu^2} + \frac{1}{\lambda^2\mu^2}\right], \qquad (125)$$

whence, on integration

$$\psi_1 = \frac{(\lambda^2 - \mu^2)}{64a^2(\log R)^2}\left[\tfrac{1}{4}\{\log(\lambda\mu)\}^2 + \tfrac{1}{4}\log(\lambda\mu) + A + B\lambda\mu + \frac{E}{\lambda^2\mu^2} + \frac{F}{\lambda\mu}\right], \qquad (126)$$

A, B, E, F, denoting constants to be determined. On introduction of the boundary conditions for ψ_1, we obtain the following system of equations :—

$$A + Ba^2 + \frac{E}{a^4} + \frac{F}{a^2} = -(\log a)^2 - \tfrac{1}{2}\log a,$$

$$A + BR^2 + \frac{E}{R^4} + \frac{F}{R^2} = -(\log R)^2 - \tfrac{1}{2}\log R,$$

$$B - \frac{2E}{a^6} - \frac{F}{a^4} = -\frac{\log a}{a^2} - \frac{1}{4a^2},$$

$$B - \frac{2E}{R^6} - \frac{F}{R^4} = -\frac{\log R}{R^2} - \frac{1}{4R^2}.$$

For simplicity assume the radius $a = 1$; the final result may then be expressed in the approximate form

$$\psi = -U\sin\theta\frac{r\log r}{\log R} - \frac{U^2\sin 2\theta}{16\nu}\left[\begin{array}{l}\dfrac{r^2(\log r)^2}{(\log R)^2} + \dfrac{r^2\log r}{2(\log R)^2} - \dfrac{r^4}{R^2}\left\{\dfrac{1}{\log R} + \dfrac{1}{4(\log R)^2}\right\} \\ \qquad + r^2\left\{\dfrac{1}{2\log R} + \dfrac{1}{4(\log R)^2}\right\} - r^2 - \dfrac{1}{r^2} + 2\end{array}\right]. \quad (127)$$

It will be observed that the anomalous form of the solution has now partially disappeared. Thus, with R infinite and r finite, there remain the terms

$$\psi = \frac{U^2\sin 2\theta}{16\nu}\left(r^2 + \frac{1}{r^2} - 2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

In the light of the present method the expression (128) may be viewed as a second "approximation" to the true solution in the vicinity of the cylinder; but the velocity at infinity can only be adjusted by inclusion of the anomalous members. A consideration of the symmetry of (128) shows that the resistance of the cylinder due to the terms already developed, vanishes.

The influence of the second term of the expansion for a translating cylinder is illustrated by a comparison between figs. 2, 6A, 6B. The first has already been described in § 2, and gives representative stream-lines corresponding to constant values of

$$\frac{\psi}{U} = -\sin\theta\frac{r\log r}{\log R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

on the assumption that $\log_e R = 5$ (*i.e.*, $R = 148 \cdot 4$), and that the radius of the inner cylinder is unity.

The second diagram (fig. 6A) indicates the system (ψ/U = const.), equivalent to the modified solution (127). In order to conform with the assumptions made, the ratio U/ν must be regarded as small; for convenience, the value taken is $U/\nu = 0 \cdot 1$, in any consistent system of units. It has already been emphasised (*see* § 2) that when R is selected

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only moderately large, the boundary conditions will not be accurately satisfied. The discrepancies can, however, be reduced to lie within any prescribed limits by taking R sufficiently large. A further drastic modification in the stream pattern accompanies this increase of R. The effect is illustrated in fig. 6B, in which R has been assumed increased to $26 \cdot 8 \times 10^{42}$ (corresponding to $\log_e R = 100$). Here, only those curves

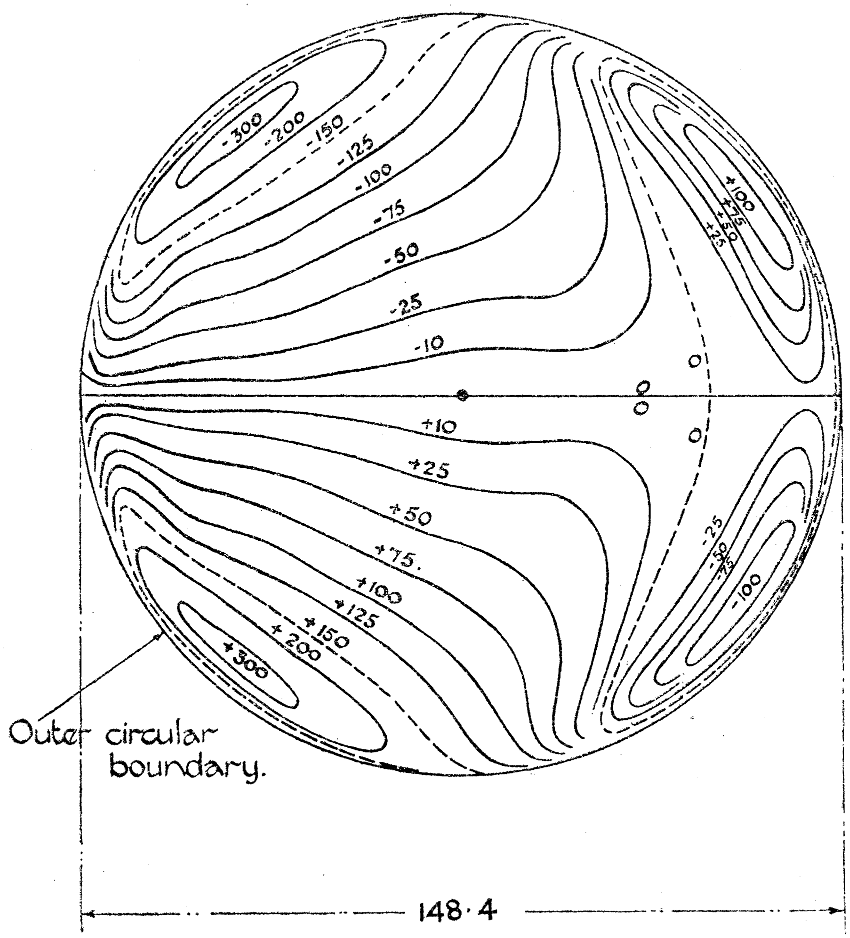

FIG. 6A.—Extended solution for the type of flow described in fig. 2, in which the outer radius $R = 148 \cdot 4$. The curves include the first two terms of the U/ν expansion, and are drawn for constant values of Ψ/U (see equation 127 of text). The fixed inner cylinder is of unit radius, and $U/\nu = 0 \cdot 1$ (in any consistent units). Maximum discrepancies of $0 \cdot 2$ U are allowed over the boundaries.

representing the highest order of value of ψ/U, and $\psi/U = 0$, can conveniently be illustrated; the lower order curves being packed between the curve $\psi/U = 0$ and the outer boundary. For this case the derivatives over the outer boundary ($r = R$) become

$$\frac{\partial \psi}{\partial r} = -1 \cdot 01 U \sin \theta,$$

$$\frac{1}{r} \frac{\partial \psi}{\partial \theta} = -U \cos \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

furnishing maximum discrepancies not exceeding 1 per cent. of U in the velocity components parallel to the axes. Discrepancies of the same order will be incurred over the inner cylinder $r = 1$.

The diagrams point to the somewhat remarkable conclusion that at the present immature, and not necessarily convergent, stage of the expansion, eddies are formed

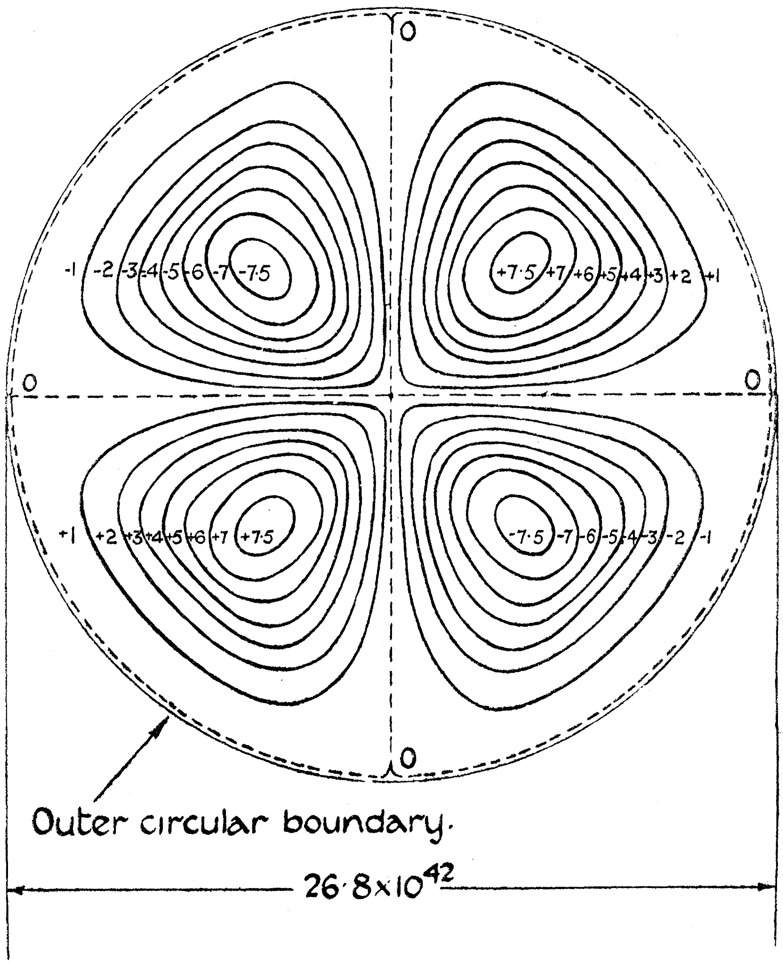

FIG. 6B.—Extended solution for type of flow described in fig. 6A, with the outer radius R increased to $26 \cdot 8 \times 10^{42}$. Maximum discrepancies of $0 \cdot 01$ U are here allowed over the boundaries. The curves are drawn for constant values of $\Psi/U \times 10^{-81}$. The lower order curves are packed between $\Psi = 0$ and the outer boundary.

both behind, and in front of, the cylinder. It is not unreasonable to suppose, however, that the leading pair of eddies *might* eventually become eliminated as the expansion proceeds to higher powers of U/ν, and that the trailing pair *might* localize themselves in the immediate wake of the cylinder. However, an extension of the analysis to the third and higher functions promises to prove exceedingly complex, since great precautions are necessary in this anomalous problem to ensure that all significant terms are retained in the expansion.

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It will be evident that parallel results might be obtained for cylinders of other cross-sectional shapes, not extending to infinity. For a system of confocal ellipses ξ = const., specified by an orthogonal transformation of type

$$\lambda \equiv x + iy = \cosh(\xi + i\eta),$$

the appropriate initial (anomalous) function is

$$\psi = -\frac{U\xi \sinh \xi}{\beta} \sin \eta,$$

in which β is assumed indefinitely large.

The most convenient further procedure is, clearly, transformation of equation (119) to the curvilinear variables ξ and η. Solutions analogous to (127) might thus be deduced both for an elliptic cylinder and for the degenerate form, the flat plate.

It is probable, however, that a more profitable line of enquiry centres in the application to simple problems in which anomalous features are absent. From the standpoint of general interest the case which, no doubt, claims priority, is that discussed in § 12, where a cylinder both translates and rotates in the presence of a wall. When $S = 0$, the solution (105) reduces to a particularly simple form. It must be remembered, however, that even with the simplest contact problems enumerated in § 11, the difficulties entailed in a purely analytical application of the class variable method will prove considerable. For this reason, it is thought desirable to limit the present paper to the preliminary stages now reached, and to reserve the extension of particular solutions for a future contribution.

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